Exercise	1	2	3	4	5	6	Total
Points							

Universidad Carlos III de Madrid

Department of Economics **Mathematics I Final Exam** **June 27th 2019**

The exam begins at: 9:00 a.m. time: 2 hours.

LAST NAME: **FIRST NAME:**

ID: **DEGREE:** **GROUP:**

(1) **Consider the function** $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$**. Then:**

(a) find its domain and its asymptotes.

(b) find the intervals where $f(x)$ increases and decreases, its local and global extrema and range (or image). Draw the graph of the function.

0.3 points part a); 0.7 points part b)

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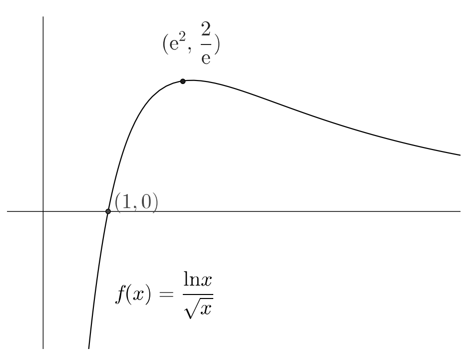
a) The domain of the given function is $(0, \infty)$. Therefore, there isn't an asymptote at $-\infty$.
Since the function is continuous on its domain we only need to study its possible vertical asymptotes at 0^+ :
$\lim_{x \longrightarrow 0^+} f(x) = \dfrac{\ln(0^+)}{0^+} = \dfrac{-\infty}{0^+} = -\infty$; thus, the function has a vertical asymptote at $x = 0^+$.
Since $\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} \dfrac{\ln(x)}{\sqrt{x}} = \dfrac{\infty}{\infty}$ =(applying L'Hopital)= $\lim_{x \longrightarrow \infty} \dfrac{1/x}{1/2\sqrt{x}} = \lim_{x \longrightarrow \infty} \dfrac{2}{\sqrt{x}} = 0$
we know that the function has a horizontal asymptote $y = 0$ at ∞.

b) Since $f'(x) = \dfrac{(1/x)\sqrt{x} - (1/2\sqrt{x}) \ln x}{x}$, we can deduce that:
f is increasing $\iff f'(x) > 0 \iff (1/x)\sqrt{x} - (1/2\sqrt{x}) \ln x > 0 \iff$ (multiplying by $2\sqrt{x}$) $\iff 2 - \ln x > 0 \iff 2 > \ln x \iff x < e^2$; so, f is increasing on $(0, e^2]$. Analogously, f is decreasing on $[e^2, \infty)$. We can deduce that f attains a local and global maximum at $x = e^2$ but it hasn‘t got a global or a local minimum.
Since $f(e^2) = \dfrac{\ln(e^2)}{\sqrt{e^2}} = \dfrac{2}{e}$, $\lim_{x \longrightarrow 0^+} f(x) = -\infty$, $\lim_{x \longrightarrow \infty} f(x) = 0$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $(-\infty,\, f(e^2)] = (-\infty, \dfrac{2}{e}]$
Thus, the graph of f will have an appearance approximately similar to:



(2) **Given the implicit function $y = f(x)$, defined by the equation $\ln(2x+y)+3y = 10x+3$ in a neighbourhood of the point $x = 0, y = 1$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 0$.

(b) sketch the graph of the function f next to the point $x = 0$. Use the tangent line to the graph of $f(x)$ to obtain the approximate values of $f(-0.1)$ and $f(0.2)$. Could you justify if any of the approximate values are rounded up or down?

0.6 points part a); 0.4 points part b)

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a) First of all, we calculate the first-order derivative of the function:

$$\frac{2+y'}{2x+y} + 3y' = 10$$

evaluating at $x = 0,\ y(0) = 1$ we obtain:

$2 + 4y'(0) = 10 \Longrightarrow y'(0) = f'(0) = 2$

Then the equation of the tangent line is: $y = P_1(x) = 1 + 2(x-0)$ or $y = 1 + 2x$.

Secondly, we calculate the second-order derivative of the function:

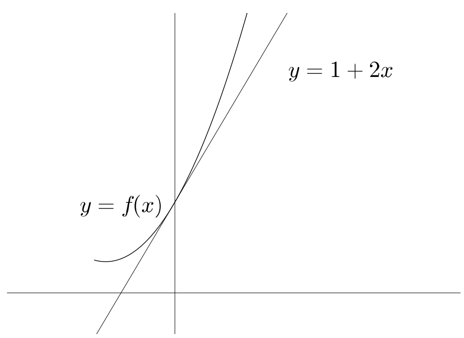
$$\frac{y''(2x+y) - (2+y')^2}{(2x+y)^2} + 3y'' = 0$$

evaluating at $x = 0,\ y(0) = 1,\ y'(0) = 2$ we obtain

$4y''(0) - 16 = 0 \Longrightarrow y''(0) = f''(0) = 4$

Therefore the second-order Taylor Polynomial is: $y = P_2(x) = 1 + 2x + \frac{4}{2}x^2 = 1 + 2x + 2x^2$.

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be:



On the other hand, the first order approximation will be:

$f(-0.1) \approx P_1(-0.1) = 0.8$; $f(0.2) \approx P_1(0.2) = 1.4$

Since f is convex in a neighbourhood of $x = 0$, because $f''(0) > 0$, the approximation values of f will be rounded down in both cases.

(3) **Let** $C(x) = C_0 + ax + 2x^2$ **be the cost function and** $p(x) = 100 - x$ **the inverse demand function of a monopolistic firm, being** $x \geqslant 0$ **the number of units produced of certain goods. Then:**

(a) Calculate a y C_0 such that the production $x = 15$ maximizes the profit.

(b) Suppose now that $C_0 = 200$. Calculate a such that the minimum average cost is 50 monetary units.

0.5 points part a); 0.5 points part b)

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a) First of all, we calculate the profit function.
$B(x) = (100 - x)x - (C_0 + ax + 2x^2) = -3x^2 + (100 - a)x - C_0$
Secondly we calculate the first and second order derivative of B :
$B'(x) = -6x + 100 - a; B''(x) = -6 < 0$
we see that B has a unique critical point at $x = \dfrac{100 - a}{6} = 15$ when $a = 10$ and, since B is a concave function, the critical point is the unique global minimizer.
C_0 can take any value.

b) The average cost function is $C_m(x) = \dfrac{C(x)}{x} = \dfrac{200}{x} + a + 2x$.
If we calculate its first and second order derivatives:
$C_m'(x) = \dfrac{-200}{x^2} + 2; C_m''(x) = 2\dfrac{200}{x^3} > 0$
we can see that $C_m'(x) = \dfrac{-200}{x^2} + 2 = 0 \Longleftrightarrow x^2 = 100$,
then $x = 10$ is the unique critical point of the function $C_m(x)$.
Since the function is convex, the critical point is the unique global minimizer.
Therefore, the production that minimizes the average cost is $x = 10$ and the minimum average cost is 50: $C_m(10) = \dfrac{200}{10} + a + 20 = 50$. Then $a = 10$.

(4) **Let** $f(x) = \begin{cases} a & si\ x = -1 \\ \sqrt[3]{x} & si\ -1 < x < 8 \\ b & si\ x = 8 \end{cases}$ **be a piece-wise defined function on the interval** $[-1, 8]$.

Then:

(a) Calculate a and b such that $f(x)$ satisfies the hypothesis (or initial conditions) of Weierstrass' Theorem on the given interval.

(b) Calculate a, b such that the thesis (or conclusion) of Weierstrass' Theorem is satisfied.
Hint for part a) and b): state Weierstrass' Theorem.

(c) Find a and b such that $f(x)$ will be increasing on the interval $[-1, 8]$.
Could $f(x)$ be increasing and discontinuous?
Hint for part c) give some real values for: a, b.
0.3 points part a); 0.4 points part b); 0.3 points part c)

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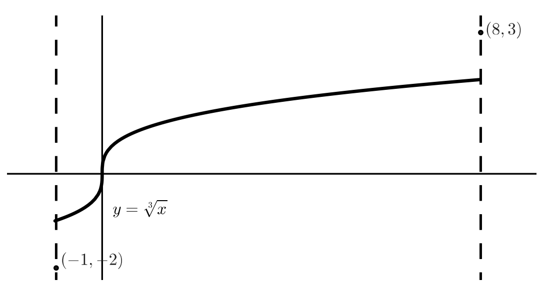
a) We need to impose continuity at $x = -1^+$ and $x = 8^-$.
Since $\lim\limits_{x \longrightarrow -1^+} f(x) = \sqrt[3]{-1} = -1, f(-1) = a$,
we can deduce that the function will be right-sided continuous at -1 if $a = -1$.

On the other hand, since $\lim\limits_{x \longrightarrow 8^-} f(x) = \sqrt[3]{8} = 2, f(8) = b$, we can deduce that the function will be left-sided continuous at 8 when $b = 2$.

b) Obviously, the thesis of Weierstrass' Theorem is satisfied when the range of the function has maximum and minimum.
The range of the function is the set $(-1, 2) \cup \{a, b\}$. Thus, the conclusion of the theorem is satisfied when $\min(a, b) \leq -1$ and $2 \leq \max(a, b)$.

c) The function will be increasing in its domain if $a \leq -1$ and $2 \leq b$.
But $f(x)$ could be discontinuous and increasing.
For example: if $a = -2, b = 3$, the function is not continuous but is increasing, since $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.
The graph of f will have an appearance approximately, similar to:



(5) **Given the function** $h(x) = xe^{-x}$, **Then:**

(a) draw approximately the set $A = \{(x, y) : 1 \leq x \leq 2, h(x) \leq y \leq 3 - x)\}$ and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for a: Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b)

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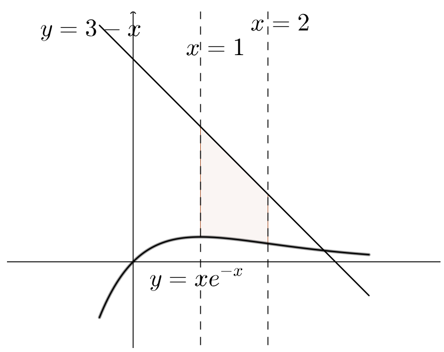
a) Beacuse the function is continuous and $h'(x) = e^{-x}(1 - x) < 0$ if $1 < x < 2$ then it is decreasing on the closed interval $[1, 2]$.
Moreover, the straight line $3 - x$ is also decreasing.
Since both functions are decreasing and $\max h(x) = h(1) = \frac{1}{e} < 1 = 3 - 2 = \min(3 - x)$ we can deduce that $h(x) < 3 - x$, for every $x\epsilon[1, 2]$.
So, the drawing of A will be, approximately, this way:



with this graph, the Pareto order describes the set in the following way:
there is no maximum, maximals$(A) = \{(x, 3 - x)) : 1 \leq x \leq 2\}$.
There is no minimum minimals$(A) = \{(x, xe^{-x}) : 1 \leq x \leq 2\}$.

b) First of all, we calculate the primitive function of f, integrating by parts:
$\int xe^{-x} = \int fg' = fg - \int f'g = x(-e^{-x}) - \int 1(-e^{-x}) = x(-e^{-x}) + \int e^{-x} = (x + 1)(-e^{-x})$
Then applying Barrow's Rule and taking into acount that $h(x) < 3 - x$,
we obtain: Area(A)$= \int_1^2 (3 - x - h(x))dx = [3x - \frac{1}{2}x^2 + (x + 1)(e^{-x})]_1^2 = \frac{3}{2} + 3\mathrm{e}^{-2} - 2\mathrm{e}^{-1}$

(6) **Given the function** $F(x) = \int\limits_1^x \frac{1}{1+t^4} dt$, **then:**

(a) Find the second order Taylor's Polynomial center at $a = 1$, of $F(x)$.

(b) Prove that $F(x) < \frac{1}{3}$ when $x > 1$. Discuss if $F(x)$ is a monotonic function and if it has an asymptote at ∞. Sketch approximately, the graph of the function $F(x)$.

Hint for part a): don't try to find $F(x)$ explicitly.

Hint for part b): find a function $g(x)$ such that $\frac{1}{1+x^4} < g(x)$, when $x \geqslant 1$.

0.4 points part a); 0.6 points part b)

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a) $F(1) = 0$, clearly.

$F'(x) = \frac{1}{1+x^4}$. then $F'(1) = \frac{1}{2}$.

$F''(x) = \frac{-4x^3}{(1+x^4)^2}$. then $F''(1) = -1$.

Thus, the second order Taylor's Polynomial center at $a = 1$, of $F(x)$ will be:

$P(x) = \frac{1}{2}(x-1) - \frac{1}{2}(x-1)^2$.

b) Since $\frac{1}{1+t^4} < \frac{1}{t^4}$, we can deduce that

$F(x) = \int\limits_1^x \frac{1}{1+t^4} dt < \int\limits_1^x \frac{1}{t^4} dt = [\frac{-t^{-3}}{3}]_1^x = \frac{-x^{-3}}{3} + \frac{1}{3}$.

and we can follow that $F(x) < \frac{1}{3}$.

Clearly, $F(x)$ is an increasing function, because has a positive derivative. Since $F(x)$, is increasing and it is bounded by $\frac{1}{3}$, has a horizontal asymptote $y = H$, where $0 < H \leq \frac{1}{3}$.

Then, the graph of $F(x)$ will have an appearance approximately, similar to: