Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Final Exam of Mathematics I** **June 19th, 2018**

Duration of the Exam: 2 hours.

SURNAMES:	**NAME:**
DNI: **GRADE:**	**GROUP:**

(1) **Let the function** $f(x) = \dfrac{x^2}{x^4+1}$**. You are asked:**

(a) Represent the graph of $f(x)$ finding previously symmetries, asymptotes, growth and decrease intervals, local and/or global extrema and range of $f(x)$.

(b) Consider the function $f(x)$ restricted to the interval $[0,1]$. Draw the graph of $f^{-1}(x)$, finding previously the domain, the range and the growth and decrease intervals of $f^{-1}(x)$.
Suggestion for b): don't try to find the the analytical expression of $f^{-1}(x)$.
0.6 points part a); 0.4 points part b).

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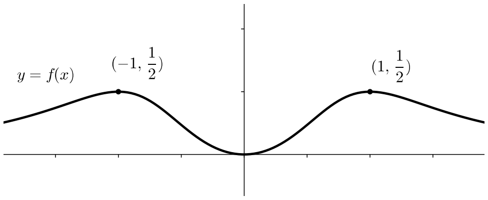
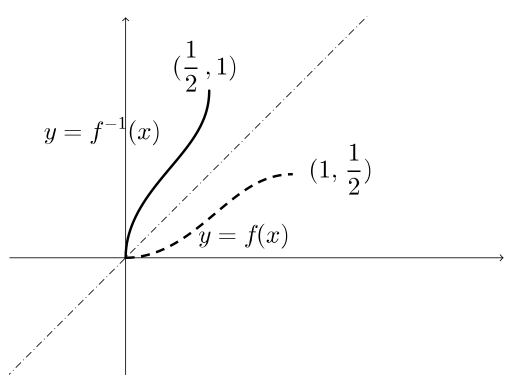
a) The domain is the set of all real numbers. Since $f(-x) = f(x)$, the function is even and its graph is symmetric with respect to the y-axis so and we only need to study the function on the interval $[0,\infty)$.
There is a horizontal asymptote at ∞, since $\lim\limits_{x\longrightarrow\infty} \dfrac{x^2}{x^4+1} = 0 \Longrightarrow \lim\limits_{x\longrightarrow-\infty} f(x) = 0$.
In order to study the monotonicity of the function we calculate the derived function:
$f\,'(x) = \dfrac{2x(x^4+1) - x^2 4x^3}{(x^4+1)^2} = \dfrac{2x-2x^5}{(x^4+1)^2} = \dfrac{2x(1-x^4)}{(x^4+1)^2}$, and we can deduce:
f is increasing on $(-\infty,-1]$ and on $[0,1]$, since, $f\,'(x) > 0$ on $(-\infty,-1)$ and $(0,1)$.
f is decreasing on $[-1,0]$ and on $[1,\infty)$, since, $f\,'(x) < 0$ on $(-1,0)$ and $(1,\infty)$.
Therefore f attains a local maximum at $x=-1$ and $x=1$, and attains a local minimum at $x=0$.
Furthermore, taking into account the symmetry of the function, its continuity and its tendency at infinity those maxima are also global.
And because $f(x) > 0$ when $x \neq 0$, $x = 0$ is also a global minimum.
Finally, we can state that the range of f is $[f(0), f(1)] = [0, \frac{1}{2}]$.
Conclusion: the graph of $f(x)$ will have an appearance, approximately, similar to the first drawing (A).



(A)



(B)

b) We know that the function $f(x)$, is continuous and increasing on $[0,1]$, with range $[0,\frac{1}{2}]$.
Therefore, its inverse function exists, it is continuous, increasing, its domain is the interval $[0,\frac{1}{2}]$ and its range the interval $[0,1]$.
Then, the graph of the function $f^{-1}(x)$ will be approximately sketched as in the second figure (B).

(2) **Let $y = f(x)$ the function defined implicitly near the point $(0, 1)$ from the equation: $y^2 - 3xy + x = 1$. It is requested:**

(a) Find the first and second derivatives of the function f at the point $x = 0, y = 1$.

(b) Find the tangent line and the Taylor polynomial of second order of the function f at the point $(0, 1)$. Represent the graph of that function near that point.

0.4 points part a); 0.6 points part b).

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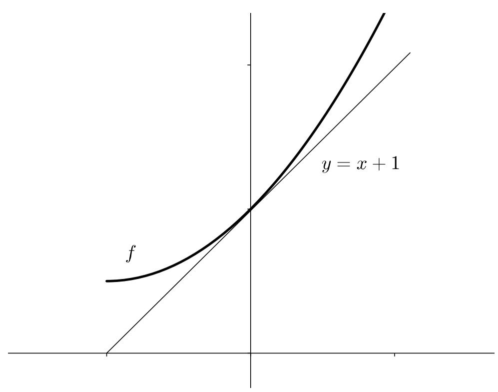
a) First of all, we compute the first derivative of the function:
$2yy' - 3y - 3xy' + 1 = 0$.
substituting $x = 0, y = 1$ in the previous equation, we obtain:
$2y' - 3 + 1 = 0 \Longrightarrow y' = 1$.
Analogously, we compute the second derivative of the function:
$2(y')^2 + 2yy" - 3y' - 3y' - 3xy" = 0$.
substituting $x = 0, y = 1, y' = 1$ in this last equation, we obtain:
$2 + 2y" - 6 = 0 \Longrightarrow 2y" = 4 \Longrightarrow y" = 2$.

b) The equation of the tangent line will be:
$y = 1 + x$.
The equation of second order Taylor's polynomial will be:
$y = 1 + x + \frac{1}{2}2x^2 = 1 + x + x^2$
Thus, the implicit function will be convex and increasing on a neighbourhood of the point $x = 0$,
So, the drawing of the function will be, approximately like the following:



(3) **Let $C'(x) = 0.2x + 3$ and $I'(x) = -0.05x + 103$ the marginal cost and revenue functions of a monopolist firm, being $x \geqslant 0$ the number of units produced of a certain commodity. You are asked to:**

(a) Decide the production that maximizes the profit. For this level of production, which will be the saving of cost (approximately) of producing one unit less?

(b) Find the level of production that minimizes the mean cost, knowing that the cost of producing 10 units is 80 monetary units. For this level of production, which will be the additional profit (approximately) of producing one unit more?

0.4 points part a); 0.6 points part b).

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a) First of all, we calculate the first and second order derived function of B :
$B'(x) = I'(x) - C'(x) = -0,05x + 103 - (0,2x + 3) = -0,25x + 100; B"(x) = -0,25 < 0$
so, we notice that B has only one critical point at $x = 400$ and, since B is a concave function then the critical point is a strictly global maximizer.
At this production level, the approximate cost savings of producing one less unit will be, 83, since $C(400) - C(399) \approx C'(400) = 83$.

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b) The total cost function is $C(x) = 0,1x^2 + 3x + C_0$.
Because $C(10) = 0,1 \cdot 10^2 + 3 \cdot 10 + C_0 = 80 \Longrightarrow C_0 = 40$,
the average cost function is $C_m(x) = \dfrac{C(x)}{x} = \dfrac{40}{x} + 3 + 0,1x$.
If we compute the first and second order derivatives of this function:
$C'_m(x) = \dfrac{-40}{x^2} + 0,1; C"_m(x) = \dfrac{80}{x^3} > 0$
we observe that $x = \sqrt{\dfrac{40}{0,1}} = 20$ is the only critical point of the function and, as this function $C_m(x)$ is convex, that critical point is the only global minimizer.
Therefore, the production that will minimize the average cost will be $x = 20$. Finally, for this production level, the approximately additional profit of increasing the production by one unit will be substituting in the average cost function, the minimum average cost will be 95, since
$B(21) - B(20) \approx B'(20) = -5 + 100 = 95$

(4) **Let a, b, c be real numbers and let's consider the following piecewise function**

$$f(x) = \begin{cases} 2ax + b & \text{si } x < 1 \\ c & \text{si } x = 1 \\ \dfrac{10}{x} + 2\ln(\frac{1+x^2}{2}) & \text{si } x > 1 \end{cases}$$ **. You are asked to:**

(a) Argue, according to the values a, b, c the continuity of the previous function on the whole real line.

(b) Argue, according to the values a, b, c the derivability of the previous function on the whole real line.

1 point

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a) For any value of a, b, c the function is continuous if $x \neq 1$.
At $x = 1$, the function is right-hand continuous if it is satisfied:
$\lim\limits_{x \longrightarrow 1^+} f(x) = f(1) \Longleftrightarrow 10 = c$
Moreover, at $x = 1$, the function is left-hand continuous if it is satisfied:
$\lim\limits_{x \longrightarrow 1^-} f(x) = f(1) \Longleftrightarrow 2a + b = 10.$
then, the function $f(x)$ is continuous at every point x when $2a + b = 10 = c$.

b) Obviously, when $x \neq 1$ our function is derivable for every a, b, c.
At the point $x = 1$, we can calculate the sided derivatives using the fact that the function is continuous when $2a + b = 10 = c$.
$f'_-(1) = \lim\limits_{x \longrightarrow 1^-} f\,'(x) = \lim\limits_{x \longrightarrow 1^-} 2a = 2a$
$f'_+(1) = \lim\limits_{x \longrightarrow 1^+} f\,'(x) = \lim\limits_{x \longrightarrow 1^+} (\dfrac{-10}{x^2} + \dfrac{4x}{1 + x^2}) = -10 + 2 = -8.$
then the function will be derviable if it is continuous and $a = -4$.
In oder words, substituting $a = -4$ in the equation $2a + b = 10 \Longrightarrow b = 18$.
Finally, the function will be derivable when $a = -4, b = 18, c = 10$.

(5) **Let A be the set limited by the graph of the function $g(x) = 10 - \dfrac{6}{3-x}$ and the segment that links the points $(4, 16)$ and $(6, 12)$. You are asked to:**

(a) Represent the set A. Find the maximal and minimal points of A.

(b) Find its area.

Suggestion for a): study the monotonicity of the function $g(x)$, as well as its concavity or convexity. Likewise, the Pareto order is given by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b).

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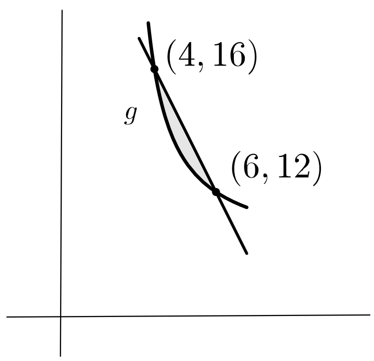
a) The function $g(x) = 10 - \dfrac{6}{3-x}$ is decreasing and convex on the interval $[4, 6]$, because $g'(x) = -6(3-x)^{-2} < 0, g"(x) = -12(3-x)^{-3} > 0$ on that interval.

Since the function $g(x)$ is convex, the segment joining the points $(4, g(4))$ and $(6, g(6))$ will be nowhere underneath the graph of $g(x)$.

Therefore, the set of points A will have an approximately graph like:



Knowing that the segment intercepts the point $(4, 16)$ and has slope equal to -2 the equation of the segment will be on the line $y = 16 - 2(x-4)$.

From the graph we can deduce that:

$\{\text{maximals}(A)\} = \{(x, y) : 4 \leq x \leq 6, y = 16 - 2(x-4)\}$.

$\{\text{minimals}(A)\} = \{(x, y) : 4 \leq x \leq 6, y = 10 - \dfrac{6}{3-x}\}$.

b) The asked area between the segment and the hyperbola will therefore be,

$\int_4^6 [(16 - 2(x-4)) - (10 - \dfrac{6}{3-x})]dx = \int_4^6 (14 - 2x + \dfrac{6}{3-x})dx = \int_4^6 (14 - 2x - \dfrac{6}{x-3})dx.$

Therefore, using Barrow's Rule we obtain the area:

$\int_4^6 (14 - 2x - \dfrac{6}{x-3})dx = [14x - x^2 - 6\ln(x-3)]_4^6 =$

$= 84 - 36 - 6\ln 3 - (56 - 16 + 0) = 8 - 6\ln 3$ area units.

(6) **Given the function** $f(x) = \dfrac{3-x}{1+x^4}$, **defined on** $[0, \infty)$, **you are asked to:**

(a) Prove that $\int_0^3 f(t)dt$ is between $1 + \frac{1}{17}$ and $4 + \frac{1}{17}$.

(b) Represent the function $F(x) = \int_0^x f(t)dt$. For it, you should find the growth and decrease intervals of $F(x)$, as well as its global maximizer, if it exists.

Suggestion for a): draw f on the interval $[0, 3]$ and consider the values $f(0), f(1), f(2)$ and $f(3)$.

Suggestion for a) and b): don't try to find explicitly $F(x)$.

0.6 points part a); 0.4 points part b).

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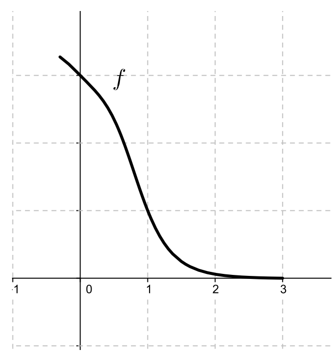
a) Obviously, $f(x)$ is a decreasing function on the interval $[0, 3]$. You can notice this state because the given function is the quotient between two positive functions, the numerator a decreasing function and the denominator an increasing one.

Or another way of knowing this is using the derived function: $f'(x) = \dfrac{(-1)(1+x^4) - 4x^3(3-x)}{(1+x^4)^2} = \dfrac{3x^4 - 12x^3 - 1}{(1+x^4)^2} = \dfrac{3x^3(x-4) - 1}{(1+x^4)^2} < 0$

Then, the graph of f will be approximately:



Moreover, since f is decreasing we have:

$f(1) < \int_0^1 f(t)dt < f(0), \quad f(2) < \int_1^2 f(t)dt < f(1), \quad f(3) < \int_2^3 f(t)dt < f(2).$

And adding together these inequalities we obtain:

$f(1) + f(2) + f(3) < \int_0^3 f(t)dt < f(0) + f(1) + f(2)$. That is, $1 + \frac{1}{17} < \int_0^3 f(t)dt < 4 + \dfrac{1}{17}$

b) $F'(x) = f(x)$, It is a positive function on $[0, 3)$ and negative on $(3, \infty)$. Thus, $F(x)$ is an increasing function on $[0, 3]$ and decreaing on $[3, \infty)$. Therfore, the function $F(x)$ attains its global maximum at the point $x = 3$. So, the graph of $F(x)$ will be approximately: