Universidad Carlos III de Madrid

Exercice	1	2	3	4	5	6	Total
Points							

Departament of Economics **Mathematics I Final Exam** **23 June 2017**

Time: 2 hours.

LAST NAMES:		**NAME:**
ID NUMBER:	**Degree Programme:**	**Group:**

(1) **Consider the function** $f(x) = e^{Q(x)}$, **where** $Q(x) = \dfrac{x^2}{x-1}$**.**

(a) Draw the graph of $f(x)$ finding previously the domain, the asymptotes, the increasing and decreasing intervals of $f(x)$, its local/global extrema and the range of $f(x)$.

(b) Let $f(x)$ be defined on the interval $[2, \infty)$. Draw the graph of $f^{-1}(x)$, finding previously the domain, the range, and the increasing and decreasing intervals of $f^{-1}(x)$.

Hint for b: solve it graphically, don't try to find the equation of $f^{-1}(x)$.

0.6 points part a); 0.4 points part b).

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a) El domain of f is the set of the real numbers, except the point $x = 1$.

There is a vertical asymptote at $x = 1^+$, because $\lim\limits_{x\longrightarrow 1^+} \dfrac{x^2}{x-1} = \infty \Longrightarrow \lim\limits_{x\longrightarrow 1^+} f(x) = \infty$.

Also, $\lim\limits_{x\longrightarrow 1^-} \dfrac{x^2}{x-1} = -\infty \Longrightarrow \lim\limits_{x\longrightarrow 1^-} f(x) = 0$.

There can be no more vertical asymptotes, as the function is continuous when $x \neq 1$.

On the other hand, there is a horizontal asymptote in $-\infty$, as $\lim\limits_{x\longrightarrow -\infty} \dfrac{x^2}{x-1} = -\infty \Longrightarrow \lim\limits_{x\longrightarrow -\infty} f(x) = 0$.

Analogously, as $\lim\limits_{x\longrightarrow \infty} \dfrac{f(x)}{x} = \frac{\infty}{\infty} =$ (by the L'Hopital's rule)

$= \lim\limits_{x\longrightarrow \infty} f(x)\dfrac{x^2-2x}{(x-1)^2} = \infty \Longrightarrow$ it doesn't exist horizontal nor oblique asymptotes in ∞.

About the monotonicity of the function, we compute the derivative and find that, if $x \neq 1$:

$f\,'(x) = f(x)\dfrac{x^2-2x}{(x-1)^2}$, so we deduce that:

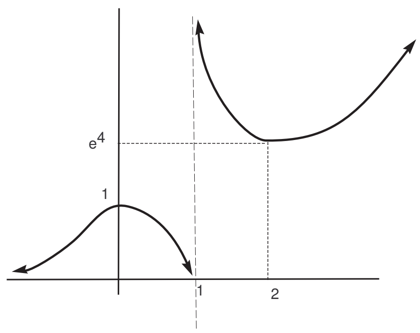
f is increasing on $(-\infty, 0]$ and on $[2, \infty)$, as $f\,'(x) > 0$ on $(-\infty, 0)$ and on $(2, \infty)$.

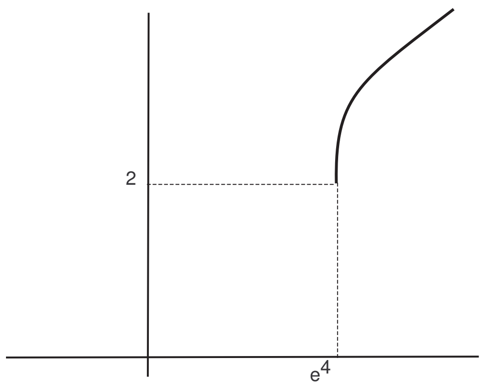
f es decreasing on $[0, 1)$ and on $(1, 2]$, as $f\,'(x) < 0$ on $(0, 1)$ and on $(1, 2)$.

So f has a local maximum at $x = 0$ and a local minimum at $x = 2$.

For that reason, its range will be $(0, 1] \cup [e^4, \infty)$.

Finally, the graph of the function $f(x)$ will be, approximately, like the first figure:





b) We begin with the function $f(x)$, continuous and increasing on $[2, \infty)$ and with range $[e^4, \infty)$. So, its inverse function is continuous and increasing, and this inverse function has as domain the interval $[e^4, \infty)$ and its range is the interval $[2, \infty)$.

Finally, the graph of the function $f^{-1}(x)$ will be, approximately, the second figure.

(2) **Let $y = f(x)$ be the function defined in a implicit way near the point $(2,1)$ by the equation:** $4xy - (x^2 + y^2) = 3$**.**

(a) Find the first and second derivatives of the function f at the point $x = 2, y = 1$.

(b) Find the tangent line and the second order Taylor's polynomial of the function f at the point $(2,1)$. Draw the graph of f near that point.

0.4 points part a; 0.6 points part b

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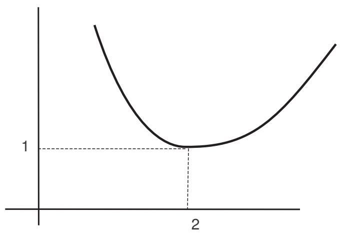
a) First of all, we derivate the equation:
$4(y + xy') - 2(x + yy') = 0$.
Substituting on that equation $x = 2, y = 1$ we obtain:
$4 + 8y' - 4 - 2y' = 0 \Longrightarrow y' = 0$.
Derivating again the equation without the substitutions:
$4(2y' + xy") - 2(1 + (y')^2 + yy") = 0$.
Substituting on the last equation $x = 2, y = 1, y' = 0$ we obtain:
$8y" - 2(1 + y") = 0 \Longrightarrow 6y" = 2 \Longrightarrow y" = \frac{1}{3}$.

b) The tangent line will have as equation:
$y = 1$.
The Taylor polynomial of order 2 will have as equation:
$y = 1 + \frac{1}{2}(\frac{1}{3})(x - 2)^2$.
For that reason, the implicit function will have a local minimum near the point $x = 2$, and its graph will be, approximately, this way:



(3) **Let** $C(x) = \sqrt{x^2 - 2x + 4}$ **be the cost function for a monopolist firm, where** $x \geq 0$ **represents the quantity in kilograms of the output.**

(a) Find the equation of the tangent line to $C(x)$ in $x = 2$, and compute an approximation of the value of $C(\ 2,1)$.

(b) Let's suppose now that the new cost function is $C_1(x) = f(C(x))$, where $f(x)$ is an increasing and derivable function such that $f(2) = 1, f\ '(2) = 3$. Calculate, for the new cost function, the equation of the tangent line to $C_1(x)$ in $x = 2$, and find an approximation to the value of $C_1(\ 2,1)$.
Have the marginal costs increased or decreased in $x = 2$, with respect to part a)?

0.4 points part a; 0.6 points part b

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a) First of all, $C'(x) = \dfrac{x-1}{\sqrt{x^2-2x+4}}$, so $C'(2) = \frac{1}{2}$.
On the other hand, as $C(2) = 2$, the equation of the tangent line will be:
$y = 2 + \frac{1}{2}(x - 2)$
Now, approximating $C(2,1)$ by the tangent line, we obtain:
$C(2,1) \approx 2 + \frac{1}{2}(2,1 - 2) = 2,05$ monetary units.

b) First of all, $C_1'(x) = f\ '(C(x)).C'(x)$, so $C_1'(2) = f\ '(2).C'(2) = \frac{3}{2}$.
On the other hand, as $C_1(2) = f(2) = 1$, the equation of the tangent line will be:
$y = 1 + \frac{3}{2}(x - 2)$
Now, approximating $C_1(2,1)$ by the tangent line, we obtain:
$C_1(2,1) \approx 1 + \frac{3}{2}(2,1 - 2) = 1,15$ monetary units.
Obviously, the marginal costs have increased in $x = 2$, as they have changed its value before from $\frac{1}{2}$, to have a value now of $\frac{3}{2}$.

(4) **Let a, b be real numbers and consider the following piecewise function**

$$f(x) = \begin{cases} ae^{4x} - be^{-4x} & \text{si } x < 0 \\ 0 & \text{si } x = 0 \\ x + \ln(1 + 2ax + 2bx) & \text{si } x > 0 \end{cases}$$

(a) Discuss, depending on the values of $a, b > 0$, the continuity of the function on the real line.

(b) Discuss, depending on the values of $a, b > 0$, the derivability of the function on the real line.

1 point

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a) For any value $a, b > 0$ the function is continuous if $x \neq 0$.

Moreove, at $x = 0$, the function is continuous by the left if it is verified that:

$\lim_{x \longrightarrow 0^-} f(x) = f(0) \Longleftrightarrow a - b = 0 \Longleftrightarrow a = b$.

On the other hand, the function is continuous at 0^+ for any $a, b > 0$.

So it is satisfied that $f(x)$ is continuous in every x when $a = b > 0$.

b) Of course, when $x \neq 0$ the previous function is derivable for any $a, b > 0$.

With respect to the point $x = 0$, let's compute the lateral derivatives, using that the function is continuous in such point when $a = b$.

$f'_-(0) = \lim_{x \longrightarrow 0^-} f'(x) = \lim_{x \longrightarrow 0^-} 4ae^{4x} + 4be^{-4x} = 4(a + b)$

$f'_+(0) = \lim_{x \longrightarrow 0^+} f'(x) = \lim_{x \longrightarrow 0^+} (1 + \dfrac{2(a + b)}{1 + 2ax + 2bx}) = 1 + 2(a + b)$.

So the function will be derivable at every point when $a = b, 2(a + b) = 1$.

In other words, when $a = b = \frac{1}{4}$

(5) **Let's consider the set of points A on the plane, bounded by the graphs of the functions** $y = \dfrac{1}{1+x}, y = -e^{-x}$ **and the lines** $x = 0,\ x = 1$**.**

(a) Draw the set A and find the maximal, minimal, maximum and minimum points of A, if they exist.

(b) Calculate the area of the region A.

Hint for a: Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

Hint for b: Don't express the value of the area in decimal numbers.

0.6 points part a; 0.4 points part b

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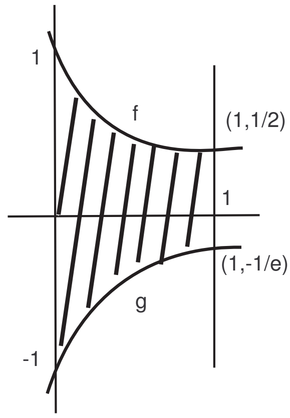
a) The function $f(x) = \dfrac{1}{1+x}$ is positive and decreasing on the interval $[0, 1]$, and the function $g(x) = -e^{-x}$ is negative and increasing on the same interval (it is enough to check that $g'(x) = e^{-x} > 0$).

Moreover, as $f(0) = 1 > -1 = g(0), f(1) = \frac{1}{2} > -\frac{1}{e} = g(1)$, the set A will have a shape, approximately, in this way:



Obviously, by the drawing you can deduce that

$\{\text{maximals}(A)\} = \{(x, y) : 0 \leq x \leq 1, y = \dfrac{1}{1+x}\} \Longrightarrow$ maximum(A) does not exist.

$\{\text{minimals}(A)\} = \{(0, -1)\} = \{\text{ minimum}(A)\}$.

b) The area of interest is the one below the rational function and above the exponential function, bounded by the vertical lines $x = 0, x = 1$.

The area is, for that reason: $\int_0^1 (\dfrac{1}{1+x} - (-e^{-x}))dx = \int_0^1 (\dfrac{1}{1+x} + e^{-x})dx$

Integrating in a direct way:

$\int (\dfrac{1}{1+x} + e^{-x})dx = \ln(1 + x) - e^{-x}$

So, appliying the Barrow's rule, you can obtain that the value of the area is:

$\int_0^1 (\dfrac{1}{1+x} + e^{-x})dx = [\ln(1 + x) - e^{-x}]_0^1 = \ln 2 - e^{-1} - (0 - 1) =$

$= 1 - e^{-1} + \ln 2$ area units.

(6) **Given the function** $f(x) = \dfrac{x^3}{1+x^4}$, **defined on** $[0,2]$, **we ask:**

(a) Find the area bounded between the graph of such function, the horizontal axis and the vertical line $x = 2$.

(b) Calculate approximately, using the Taylor polynomial of order 2 of $F(x) = \displaystyle\int_1^x f(t)dt$ centered at $x = 1$, the area bounded between the graph of such function f, the horizontal axis and the vertical lines $x = 1$ and $x = 1,1$.

0.4 points part a; 0.6 points part b

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a) $f(x)$ is a continuous and positive function on the interval $[0,2]$.

So the area of interest is equal to the integral $\displaystyle\int_0^2 \frac{x^3}{1+x^4}dx$.

The primitive of f is $\int \dfrac{x^3}{1+x^4} = \frac{1}{4}\ln(1+x^4)$.

So, the area of interest is, by Barrow's rule, equal to:

$\displaystyle\int_0^2 \frac{x^3}{1+x^4}dx = [\frac{1}{4}\ln(1+x^4)]_0^2 = \frac{1}{4}\ln(17)$ area units.

b) As the Taylor polynomial, centered at $x = 1$, of the function $F(x)$ is:

$P(x) = F(1) + F'(1)(x-1) + \frac{1}{2}F"(1)(x-1)^2$

then, $P(1,1)$ is an approximation of $F(1,1) = \displaystyle\int_1^{1,1} f(t)dt$, the requested area.

Obviously, $F(1) = \displaystyle\int_1^1 f(t)dt = 0$,

$F'(1) =$(by the fundamental theorem of calculus)$= f(1) = \frac{1}{2}$.

And, as $f\,'(x) = \dfrac{3x^2(1+x^4) - 4x^3x^3}{(1+x^4)^2} = \dfrac{3x^2 - x^6}{(1+x^4)^2} \Longrightarrow F"(1) = f\,'(1) = \frac{1}{2}$

For those reasons, $P(x) = \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2$.

So, approximated area$= P(1,1) = \frac{1}{2}.0,1 + \frac{1}{4}.0,01 = 0,0525$ area units.