Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **24 June 2016**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = \sqrt[3]{x^2(1-x)}$**. Then:**

(a) draw the graph of the function, obtaining firstly its domain, the intervals where $f(x)$ increases and decreases, its local and/or global extrema (if they exist), asymptotes and range.

(b) consider the new function $f(x)$ only defined on the interval $[0, \frac{2}{3}]$. Sketch the graph of the function $f^{-1}(x)$, obtaining firstly its domain, the range, the intervals where this function increases and decreases and calculate its fixed points.

Hint 1: Calculate only the asymptote, if it exists, at ∞. Because at $-\infty$ it is exactly the same.

Hint 2: Don't try to calculate the analytic expression of $f^{-1}(x)$. Furthermore, the fixed points of any function are the same as its inverse.

Part (a) 0.6 points; Part (b) 0.4 points.

---

a) The domain of the function is the whole real line.

Since the function is continuous everywhere, there are no vertical asymptotes.

There is an oblique asymptote at ∞, because $\lim_{x\longrightarrow\infty} \frac{f(x)}{x} = \lim_{x\longrightarrow\infty} \sqrt[3]{(1-x)/x} = -1$.

Furthermore, $\lim_{x\longrightarrow\infty} f(x) + x = \lim_{x\longrightarrow\infty} [\sqrt[3]{x^2(1-x)} + x] = \lim_{x\longrightarrow\infty} [(\sqrt[3]{(1-x)/x} + 1)x] =$

$= \lim_{x\longrightarrow\infty} \frac{\sqrt[3]{1/x - 1} + 1}{1/x} = \frac{0}{0} =$ (using L'Hospital's Rule)

$= \lim_{x\longrightarrow\infty} \frac{(1/3)(1/x-1))^{2/3}(-1/x^2)}{-1/x^2} = \lim_{x\longrightarrow\infty} \frac{1}{3}(1/x-1)^{2/3} = 1/3$

Thus, the straight line $y = -x + \frac{1}{3}$ is the oblique asymptote at ∞.

In the same way, this line is also the oblique asymptote at $-\infty$.

Then, there are no horizontal asymptotes nor global extreme values for the function. Since the function is continuous on the whole real line, its range is also the whole real line.

In order to find the increasing/decreasing interval for the function we calculate its derived function and we see that if $x \neq 0, 1$:

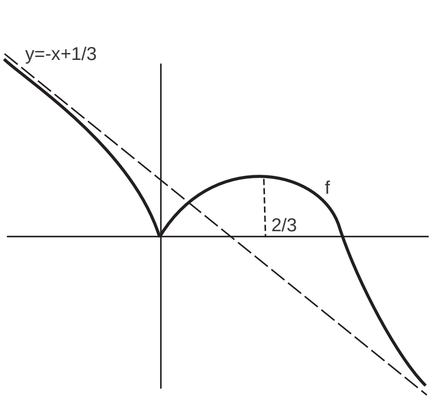
$f\,'(x) = \frac{1}{3}(x^2 - x^3)^{-2/3}(2x - 3x^2)$, and we can deduce that

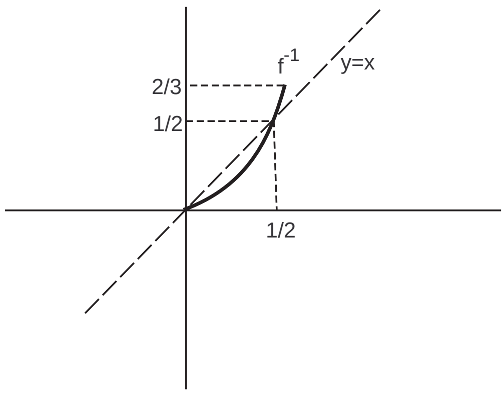
f is increasing on $[0, \frac{2}{3}]$, since $f\,'(x) > 0$ on $(0, \frac{2}{3})$.

f is decreasing on $(-\infty, 0]$ and on $[\frac{2}{3}, \infty)$, since $f\,'(x) < 0$ on $(-\infty, 0)$ and also on $(\frac{2}{3}, 1) \cup (1, \infty)$.

Therefore, f attains a local minimum at $x = 0$ and a local maximum at $x = \frac{2}{3}$.

The graph of the function, $f(x)$, will be approximately as you can see in the first figure:





b) Starting with the function $f(x)$, continuous and increasing on $[0, \frac{2}{3}]$ whose range is $[0, \sqrt[3]{4}/3]$.

We know that its inverse function is also continuous and increasing on its domain, the interval $[0, \sqrt[3]{4}/3]$ and its range will be the interval $[0, \frac{2}{3}]$.

Finally, the fixed points for the function f are $x = 0, x = \frac{1}{2}$, since $f(x) = x \iff$
$\iff (x^2(1-x))^{1/3} = x \iff x^2(1-x) = x^3 \iff 1 - x = x$ o $x = 0 \iff x = \frac{1}{2}$ o $x = 0$.

So, the graph of the inverse function $f^{-1}(x)$ is more or less as you can see in the second figure.

(2) **Let $y = f(x)$ be the implicit function defined by the equation $2y+e^{x-y} = 3$, in a neighborhood of the point $(1,1)$. Then:**

(a) find the derivative of first and second order of $f(x)$ at the point $x = 1, y = 1$.

(b) find the tangent line and the second order Taylor polynomial of f at $(1,1)$. Sketch the graph of the function around that point.

Part (a) 0.4 points; Part (b) 0.6 points

---

a) Firstly, we calculate the derivative of the equation:

$2y' + e^{x-y}(1 - y') = 0.$

and evaluate $x = 1, y = 1$ for that equation. We obtain:

$2y' + 1 - y' = 0 \Longleftrightarrow y' = -1.$

Secondly, we calculate the second derivative of the equation:

$2y" + e^{x-y}[(1 - y')^2 - y"] = 0$

and also evaluating $x = 1, y = 1, y' = -1$ for the second derived equation. We obtain:

$y" + 4 = 0 \Longrightarrow y" = -4.$

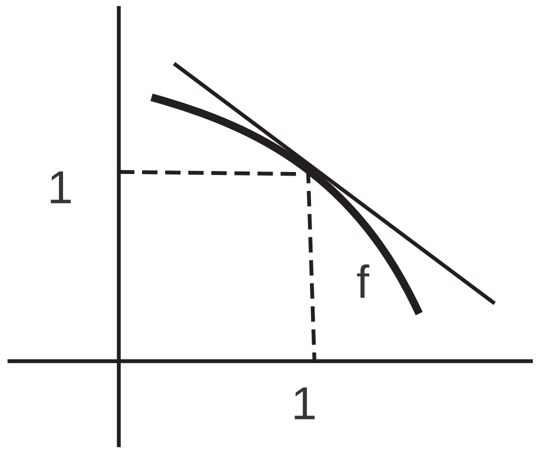
b) Then equation of the tangent line is:

$y = 1 + (-1)(x - 1) = -x + 2.$

Now, the second order Taylor's polynomial is:

$y = 1 + (-1)(x - 1) + \frac{1}{2}(-4)(x - 1)^2$

Thus, the implicit function is decreasing and concave near the point $x = 1$, and its graph is approximately like this:



(3) **Let $C(x) = 0,01ax^2 + 2x + a$ be the cost function and $p(x) = b - 2x$ the inverse demand function of a monopolistic firm. Then:**

(a) find, depending on $a > 0$, the production x_0 that minimizes the firm's average cost.

(b) suppose now that $a = 100$ and $b > 2$. Calculate, depending on b, the production x_1 that maximizes the profit of the company.

Hint 1:The production on Part (a) and (b) may or may not depend on the parameters.

1 point

---

a) Let $\dfrac{C(x)}{x} = 0,01ax + 2 + \dfrac{a}{x}$ be the average cost function.

Calculating its derived function we obtain:

$(\dfrac{C(x)}{x})' = 0,01a - \dfrac{a}{x^2} = 0 \Longleftrightarrow x^2 = 100 \Longleftrightarrow x_0 = 10$

and it doesn't depend on the parameter a.

If you notice that the average cost function is convex, since $(\dfrac{C(x)}{x})'' > 0$,

the critical point is the only global minimizer for the function.

b) Considering $a = 100$, the profit function is:

$B(x) = I(x) - C(x) = bx - 2x^2 - (x^2 + 2x + a) = -3x^2 + (b - 2)x - 100$

the critical point, x_1 of the profit function is:

$B'(x_1) = -6x_1 + b - 2 = 0 \Longleftrightarrow x_1 = (b - 2)/6.$

If we notice that the profit function is concave, since $B''(x) < 0$,

then the critical point is the only global maximizer for the profit function.

(4) **Let a, b be two real numbers and consider the following piecewise function:**

$$f(x) = \begin{cases} 4ae^{ax} & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 2\sqrt{x+b} & \text{if } x > 0 \end{cases}$$ **. Then:**

(a) show, depending on a, b, if the function $f(x)$ is continuous on the whole real line.

(b) show, depending on a, b, if the function $f(x)$ is differentiable on the whole real line.

1 point

---

a) For any value of a, the function is continuous on $x < 0$.

Furthermore, at $x = 0$, the function is continuous from the left hand side if:
$\lim_{x \longrightarrow 0^-} f(x) = f(0) \iff 4ae^0 = 2 \iff a = \frac{1}{2}$.
On the other side, the function is continuous if $x > 0$ when $b \geqslant 0$.
And in particular, f is continuous at $x = 0$ from the right side if it is satisfied
$\lim_{x \longrightarrow 0^+} f(x) = f(0) \iff 2\sqrt{b} = 2 \iff b = 1$.
Then we can say that $f(x)$ is continuous on every x when $a = \frac{1}{2}, b = 1$.

b) We all agree that if $x \neq 0$ the function is derivable if $b \geqslant 0$ since, there is a neighbourhood of any point where the function is just an exponential or square root derivable function.

Meanwhile at the point $x = 0$, we are going to calculate both sided derivatives knowing that the function is continuous at that point if $a = \frac{1}{2}, b = 1$.

$f'_-(0) = \lim_{x \longrightarrow 0^-} f'(x) = \lim_{x \longrightarrow 0^-} 4a^2e^{ax} = 4a^2e^0 = 4a^2 = 1$

$f'_+(0) = \lim_{x \longrightarrow 0^+} f'(x) = \lim_{x \longrightarrow 0^+} \dfrac{2}{2\sqrt{x+1}} = 1.$

Then the function is derivable everywhere if $a = \frac{1}{2}, b = 1$.

(5) **Consider the set of points A bounded by the graph of the functions $y = x^3, y = -xe^{x-2} + 10$ and the straight lines $x = 0$, $x = 2$. Then:**

(a) draw the set A and find, if they exist, the maximal/minimal elements, the maximum and the minimum of A.

(b) calculate the area enclosed by the set A.

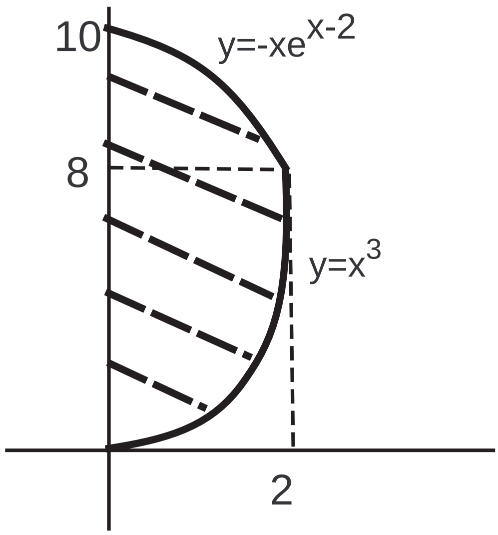
Hint for (a): Pareto order is given by: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

Hint for (b): Don't try to calculate exactly its value, You only need to indicate it.

1 point

---

a) The function $f(x) = x^3$ is positive and increasing on the interval $[0, 2]$,
and the function $g(x) = -xe^{x-2} + 10$ is decreasing on the same interval
(notice that $g'(x) = -e^{-2}e^x(x+1) < 0$) and it is positive on that interval, since $g(2) > 0$.
Furthermore, since $f(0) = 0 < 10 = g(0), f(2) = 8 = g(2)$, we can sketch the set A approximately like this:



Obviously , from the graph we can deduce
$\{\text{maximals}(A)\} = \{(x, y) : 0 \leq x \leq 2, y = -xe^{x-2} + 10\} \Longrightarrow$ and the maximum of (A) doesn't exist.
$\{\text{minimals}(A)\} = \{(0, 0)\} = \{\ \text{minimum}(A)\}$.

b) The asked area is underneath the exponential function and above the power function bounded by the vertical lines $x = 0, x = 2$.

The area is: $\int_0^2 (-xe^{x-2} + 10 - x^3)dx$

Integrating by parts:

$\int xe^x dx = xe^x - \int e^x dx = (x-1)e^x$, so

$\int(-xe^{x-2} + 10 - x^3)dx = -e^{-2}\int xe^x dx + 10x - \int x^3 dx = -e^{-2}(x-1)e^x + 10x - x^4/4$.

And, using Barrow's Rule, we obtain the area:

$\int_0^2 (-xe^{x-2} + 10 - x^3)dx = [-e^{-2}(x-1)e^x + 10x - x^4/4]_0^2 = -e^{-2}(2-1)e^2 + 20 - 2^4/4 - e^{-2} =$
$= -1 + 20 - 4 - e^{-2} = 15 - e^{-2}$ area units.

(6) **Given the function** $f(x) = \dfrac{x}{1+x^2}$, **defined on** $[0, 2]$, **then:**

(a) find its local and/or global extrema on the given interval.
State any theorem that you use.

(b) without the calculus of any primitive function give the best approximate value from above and below, using rational numbers, of the integral $\displaystyle\int_0^2 \frac{x}{1+x^2}dx$

Hint for (b): Use the information from part (a) and sketch the graph of f.

1 point

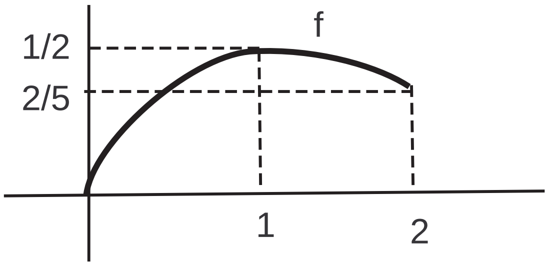
---

a) $f(x)$ is a continuous function on the closed and bounded interval $[0, 2]$, , then using Weierstrass' theorem we know that the function attains its global extreme values on that interval.

Since $f\,'(x) = \dfrac{1-x^2}{(1+x^2)^2} = 0 \Longleftrightarrow x = 1$, if $x \in [0, 2]$, we know that $x = 1$ is the only possible local extreme value of the function on $(0, 2)$.

As $f\,'(x) > 0$ if $0 < x < 1$, then f is increasing on the interval $[0, 1]$.
As $f\,'(x) < 0$ if $1 < x < 2$, then f is decreasing on the interval $[1, 2]$.
For all of that we can assume that:
i) f attains a global and local maximum at $x = 1$.
ii) As $f(0) = 0, f(2) = \frac{2}{5}$, then f attains its global and local minimum at $x = 0$.
Notice: f attains also a local minimum at $x = 2$.

b) The graph of f is approximately:



We can observe from the graph that:
i) $0 < f(x) < \frac{1}{2}$ if $x \in (0, 1) \Longrightarrow 0 < \int_0^1 \dfrac{x}{1+x^2}dx < \frac{1}{2}$.
ii) $\frac{2}{5} < f(x) < \frac{1}{2}$ if $x \in (1, 2) \Longrightarrow \frac{2}{5} < \int_1^2 \dfrac{x}{1+x^2}dx < \frac{1}{2}$.
Then, adding together these inequalities:
$0 + \frac{2}{5} < \int_0^1 \dfrac{x}{1+x^2}dx + \int_1^2 \dfrac{x}{1+x^2}dx = \int_0^2 \dfrac{x}{1+x^2}dx < \frac{1}{2} + \frac{1}{2} = 1.$
we obtain that $\frac{2}{5}$ is the best approximation of the integral from below the curve.
An in the same way, 1 is the best approximation of the integral from above the curve.