Universidad Carlos III de Madrid

Ejercicio	1	2	3	4	5	6	Total
Puntos							

Department of Economics **Matemathics I. Final Exam** **26 June 2014**

Time: 2 hours.

LAST NAMES:		**NAME:**
DNI:	**Degree programme:**	**Group:**

(1) **Consider the function f(x)=** $\dfrac{e^x}{x-1}$**.**

(a) Find the domain, the range, the asymptotes, the increasing and decreasing intervals of $f(x)$ and draw its graph.

(b) Let $f(x)$ be defined on $(1,2]$.
Find the domain, the range, the asymptotes, the increasing and decreasing intervals of $f^{-1}(x)$ and draw its graph.
Hint: Solve it graphically, don't try to find the equation of $f^{-1}(x)$.
1 point

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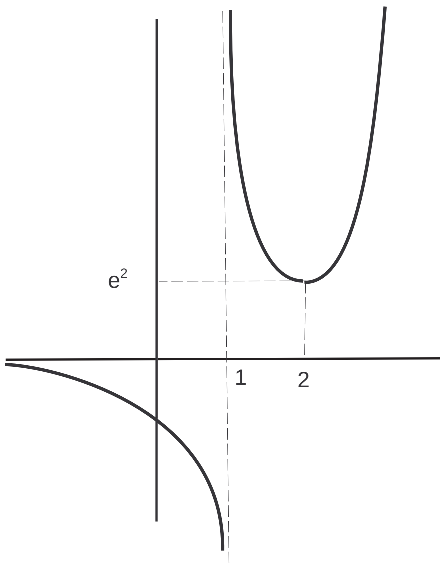
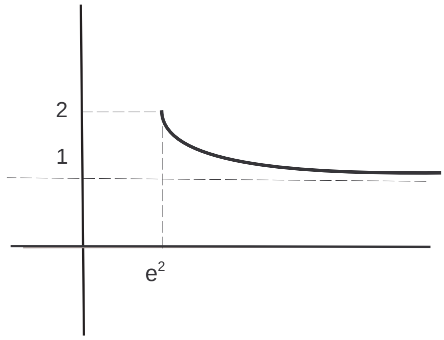
a) The domain of f is the set of real numbers except $x=1$, because at that point the denominator is equal to zero.
There is a vertical asymptote at $x=1$,
$\lim\limits_{x\longrightarrow 1-} f(x) = \dfrac{e}{0-} = -\infty$, $\lim\limits_{x\longrightarrow 1+} f(x) = \dfrac{e}{0+} = \infty$.
Also, there is a horizontal asymptote at $-\infty$, $\lim\limits_{x\longrightarrow -\infty} f(x) = \frac{0}{-\infty} = 0$.
There aren't any horizontal or oblique asymptotes at ∞, because $\lim\limits_{x\longrightarrow\infty} \dfrac{f(x)}{x} = \lim\limits_{x\longrightarrow\infty} f(x) = \infty$, you can apply L'Hopital's rule in order to calculate those limits.
Since $e^x > 0$, the derivative function $f\,'(x) = \dfrac{(x-2)e^x}{(x-1)^2}$ is positive on $[2,\infty)$, f is increasing and since the derivative function is negative on $(-\infty,1)$ and $(1,2]$, f is decreasing .
Taking into acount that $f(2)=e^2$, $\lim\limits_{x\longrightarrow 1+} f(x) = \infty = \lim\limits_{x\longrightarrow\infty} f(x)$, the function is continuous on $(1,\infty)$; with $\lim\limits_{x\longrightarrow -\infty} f(x) = 0$, $\lim\limits_{x\longrightarrow 1-} f(x) = -\infty$, and is continuous on $(-\infty,0)$;
and considering its monotonicity, the range of the function is $(-\infty,0)\cup[e^2,\infty)$.
Hence, the graph of the function $f(x)$ is more or less:



b) Knowing that $f(x)$ is continuous and decreasing on $(1,2]$ and whose range is $[e^2,\infty)$ we can say that the domain of the inverse $f^{-1}(x)$ is $[e^2,\infty)$ and its range is $(1,2]$. Furthermore, the inverse function is decreasing and has a horizontal asymptote $y=1$ at ∞, so the graph of $f^{-1}(x)$ will be:



(2) **Given the function** $f(x) = e^{bx} - e^{x}$**, with** $b \neq 1$**.**

(a) Calculate the second order Taylor´s polynomial of f at $x = 0$. What are the values of b so the function is increasing around $x = 0$? What are the values of b so the function is convex around $x = 0$?

(b) Suppose $b = 2$. Calculate the increasing and decreasing intervals and the local and/or global extreme values of f . Calculte the approximate value of $f(0.1)$.

1 point

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a) In the first place, calculate the derivatives of the function:
$f\,'(x) = be^{bx} - e^{x}$
$f"(x) = b^2 e^{bx} - e^{x}$
Finding the value of the function and the two first order derivatives at $x = 0$ we get:
$f(0) = 0, f\,'(0) = b - 1, f"(0) = b^2 - 1.$
Therefore, Taylor's second order polynomial at 0 of the function is:
$P(x) = f(0) + f\,'(0)x + \frac{1}{2}f"(0)x^2 = (b-1)x + \frac{1}{2}(b^2-1)x^2.$
The function is increasing around $x = 0$ when it is $P(x)$, in other words, when $b > 1$.
The function is convex near $x = 0$ when it is $P(x)$, in other words, when $b^2 > 1 \Longleftrightarrow |b| > 1$.

b) For $b = 2$ the derivative of the function is
$f\,'(x) = 2e^{2x} - e^{x} = (2e^{x} - 1)e^{x}$
As $f\,'(x) > 0 \Longleftrightarrow (2e^{x} - 1) > 0 \Longleftrightarrow e^{x} > \frac{1}{2} \Longleftrightarrow x > \ln\frac{1}{2} = -\ln 2$
We can deduce that f is increasing in the interval $[-\ln 2, \infty)$, where $f\,'$is positive.
In the same way, f is decreasing in the interval $(-\infty, -\ln 2]$, where $f\,'$is negative.
Therefore,the function has a local and global minimum at $x = -\ln 2$.
On the other hand, $f(0.1) \simeq P(0.1) = 0.1 + \frac{3}{2}0.01 = 0.115.$

(3) **Given the cost function** $C(x) = 800 + x + 0.02x^2$ **and the inverse demand function for a monopolist firm** $p(x) = 50 - 0.05x$**.**

(a) Find the level of output x_0 which maximizes profits.

(b) Find the level of output x_1 which minimizes the average costs.

Note: Justify all your answers.

1 point

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a) Since the profit function is
$B(x) = I(x) - C(x) = 50x - 0.05x^2 - 800 - x - 0.02x^2 = -800 + 49x - 0.07x^2$
the production x_0 that maximizes the profits will be:
$B'(x_0) = 49 - 0.14x_0 = 0 \Longleftrightarrow x_0 = 350$.
And, taking into account that the profits function is concave, as $B''(x) < 0$,
the critical point will be the only global maximum of the profits.

b) Since the average cost function is $\dfrac{C(x)}{x} = \dfrac{800}{x} + 1 + 0.02x$ and its derivative is
$(\dfrac{C(x)}{x})' = -\dfrac{800}{x^2} + 0.02 = 0 \Longleftrightarrow x = 200$, its only critical point is $x_1 = 200$.
And taking into account that the average cost function is convex, as $(\dfrac{C(x)}{x})'' > 0$,
then, the critical point will be the strict global maximum point of the average costs.

(4) **Let a, b be real numbers that define the following piecewise function**

$$f(x) = \begin{cases} \dfrac{ax+3}{(1+x)^2} & \text{if } x < 0 \\ 3 & \text{if } x = 0 \\ b + \ln(x^2+1) & \text{if } x > 0 \end{cases}$$

(a) Discuss the continuity of $f(x)$ depending on the values of a, b.

(b) Discuss the differentiability of $f(x)$ depending on the values of a, b.

1 point

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a) For any value of a, b the function is discontinuous at $x = -1$, as $f(-1)$ doesn't exist.
Furthermore, at $x = 0$ it is verified that:
$\lim_{x \longrightarrow 0-} f(x) = f(0) = 3$, $\lim_{x \longrightarrow 0+} f(x) = b$
thus, for any a value, $f(x)$ is continuous at $x = 0$ when $b = 3$.
Finally, the function is continuous at any other point in its domain.

b) Obviously, at $x = -1$ the former function is not derivable since it is not continuous.
Thus, if $x \neq 0, x \neq -1$, the former function is derivable at any point.
Finally, at $x = 0$, the function is not derivable if $b \neq 3$, since it is not continuous.
Therefore, we can observe what happens at the origin when $b = 3$.

$$\lim_{x \longrightarrow 0-} f'(x) = \lim_{x \longrightarrow 0-} \frac{a(x+1)^2 - (ax+3)2(x+1)}{(1+x)^4} = a - 6$$

$$\lim_{x \longrightarrow 0+} f'(x) = \lim_{x \longrightarrow 0+} \frac{2x}{1+x^2} = 0$$

Thus, the function will be derivable at $x = 0$ when $a = 6, b = 3$.

(5) **Suppose $a > 0$, consider the set of points A in the plane, bounded by the parabola $y = ax^2$, the straight line $y = 4a(x-1)$ and the horizontal axis.**

(a) Draw the set A and using Pareto ordering find the maximal, minimal, maximum and minimum points of A, if they exist.

(b) Calculate the area of the region A.

Note: Pareto ordering is defined: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

1 point

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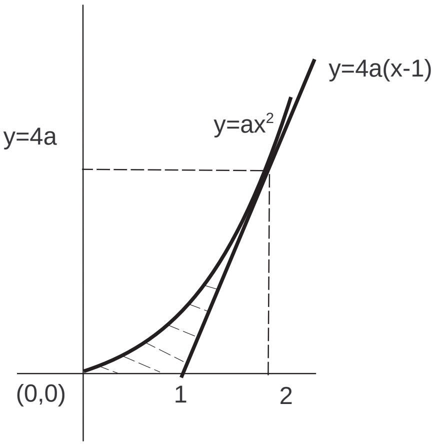
a) The line $y = 4a(x-1)$ cuts the parabola at the point $(2, 4a)$, since:
$ax^2 = 4a(x-1) \iff x^2 = 4(x-1) \iff x^2 - 4x + 4 = 0 \iff x = 2$,
and it is the tangent line on this parabola at that point, since the slope of the line is $4a$, which is equal to to the derivative of the parabola at the point $(2, 4a)$.
On the other hand, the line $y = 4a(x-1)$ intercepts the horizontal axis at the point $(1, 0)$.
Therefore, the region bounded by these functions has a shape similar to:



Obviously, from the drawing we can deduce that
$\{\text{maximals}(A)\} = \{\text{maximum}(A)\} = \{(2, 4a)\}$
$\{\text{minimals}(A)\} = \{\text{minimum}(A)\} = \{(0, 0)\}$

b) The area of interest is the one below the parabola on the interval $[0, 2]$, minus the area of the triangle formed by the points $(1, 0), (2, 0)$ and $(2, 4a)$.
Therefore, the area of interest is:
$\int_0^2 ax^2 dx - \frac{1}{2} \cdot 1 \cdot 4a = [a\frac{x^3}{3}]_0^2 - 2a = (\frac{8}{3} - 2)a = \frac{2}{3}a$ units of area.

(6) **Consider the function** $f(x) = \sqrt{x}(1 - \sqrt{x})$**:**

(a) Find the primitive function $F(x)$ of $f(x)$ such that $F(1) = 1$.

(b) Suppose that $f(x) = g(x)(1 - g(x))$ being $g(x)$ any function defined on $[0, \infty)$, continuous, increasing on its domain and such that $g(0) = 0, g(1) = 1$.
Furthermore, suppose that f is decreasing when $x > 4$ and $f(4) = -2$.
Study the intervals of increase and decrease of $F(x) = \int_0^x f(t)dt$, prove that $\lim_{x \longrightarrow \infty} F(x) = -\infty$ and sketch a graph of $F(x)$.
Hint: For the limit in b: if $x > 4, F(x) = F(4) + \int_4^x f(t)dt$.
1 point

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a) Changing of variable $t = u^2, dt = 2udu$,
we can deduce that the primitive function of f is:
$F(t) = \int \sqrt{t}(1 - \sqrt{t})dt + C = \int u(1 - u)2udu + C = \frac{2}{3}u^3 - \frac{1}{2}u^4 + C =$
$= \frac{2}{3}(\sqrt{t})^3 - \frac{1}{2}t^2 + C.$
As $F(1) = 1$, we can deduce that $1 = \frac{2}{3} - \frac{1}{2} + C \Longrightarrow C = \frac{5}{6}$.
Thus $F(x) = \frac{2}{3}(\sqrt{x})^3 - \frac{1}{2}x^2 + \frac{5}{6}$.

b) The derivative function of $F(x)$ is:
$F'(x) = f(x) = g(x)(1 - g(x))$.
Thus, we can deduce that $F'(x)$ is positive on the interval $(0, 1)$ and negative on the interval $(1, \infty)$.
Therefore, F is increasing on the interval $[0, 1]$ and decreasing on the interval $[1, \infty)$.
Finally, if $x > 4, F(x) = F(4) + \int_4^x f(t)dt < F(4) - 2(x - 4) \longrightarrow -\infty$ when $x \longrightarrow \infty$.
Thus, the graph of $F(x)$ will be similar to: