Universidad Carlos III de Madrid

Economics Department **Final Exam Mathematics I** **June 20, 2012**

Time length: 2 hours

LAST NAMES:		**FIRST NAME:**
DNI:	**Title:**	**Group:**

(1) **Let function** $f(x) = \ln(x^2 - 1)$

(a) Represent the graph of $f(x)$ after finding its domain, asymptotes, intervals at which it is increasing and decreasing, and its image.

(b) Restrict $f(x)$ to the interval $(1, \infty)$. Find the corresponding analytical expression of $f^{-1}(x)$ and draw the graph of $f^{-1}(x)$.

1 point

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a) The domain of f is restricted by the requirement $x^2 - 1 > 0$. So the domain is $(-\infty, -1) \cup (1, \infty)$. Now, f has vertical asymptotes both at $x = -1$ and at $x = 1$, since

$$\lim_{x \longrightarrow -1^-} f(x) = \lim_{x \longrightarrow 1^+} f(x) = \ln(0^+) = -\infty.$$

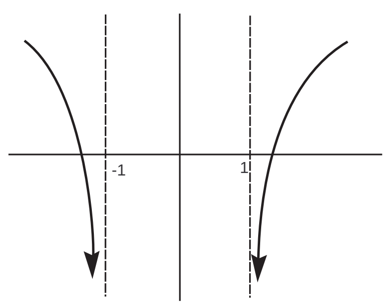
On the other hand, it has neither horizontal nor oblique asymptotes, because

$$\lim_{x \longrightarrow \pm\infty} f(x) = \infty; \lim_{x \longrightarrow \pm\infty} \frac{f(x)}{x} = (L'Hopital) = \lim_{x \longrightarrow \pm\infty} \frac{2x/(x^2-1)}{1} = 0.$$

Since $f\,'(x) = \dfrac{2x}{x^2 - 1}$, where $x^2 - 1 > 0$, we have that f is decreasing on $(-\infty, -1)$ and increasing on $(1, \infty)$.

Lastly, since $\lim_{x \longrightarrow 1^+} f(x) = -\infty$ and $\lim_{x \longrightarrow \infty} f(x) = \infty$, and f is continuous on $(1, \infty)$, its image is $\mathbb{R}$.

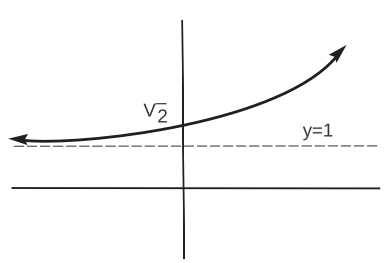
From all of this, we can draw a sketch of f as follows:



b) First we set $y = f(x) = \ln(x^2 - 1)$. Next, notice that $e^y = x^2 - 1 \Longleftrightarrow e^y + 1 = x^2$. Now, since $x > 0$, we have that $x = \sqrt{e^y + 1}$. Changing variables we conclude that

$$y = f^{-1}(x) = \sqrt{e^x + 1}$$

A sketch of $f^{-1}(x)$ is:



(2) **Let**

$$f(x) = a + x + \frac{1}{x-2},$$

where $a \geq 0$ **is a parameter.**

(a) Find the value of a such that $f(x_0) = 5$, where x_0 is the point at which f reaches its unique local maximum.

(b) Find the value of a that minimizes the integral $\int_3^4 f(x)dx$.

Remark: parts a) and b) are independent.

1 point

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a) First, let's find the first and second derivatives of f:

$$f'(x) = 1 - (x-2)^{-2};\ f''(x) = 2(x-2)^{-3}.$$

Next, we set $f'(x)$ equal to 0 to obtain the critical points:

$$f'(x) = 0 \iff 1 = (x-2)^{-2} \iff x = 1, x = 3.$$

Finally, evaluating $f''(x)$ at the critical points we find that $f''(1) < 0$. It follows that $x_0 = 1$ is a local maximum. Since $f(1) = a + 1 - 1 = 5 \Longrightarrow a = 5$.

b) Since

$$F(a) = \int_3^4 (a + x + (x-2)^{-1})dx = \left[ax + \frac{x^2}{2} + \ln(x-2)\right]_3^4 = a + \frac{7}{2} + \ln 2,$$

$F(a)$ reaches a minimum value when $a = 0$ (remember that $a \geq 0$).

(3) **Let $C'(x) = 30 + 8x$ be the marginal cost function, and $C_0 = 100$ the fixed costs of a monopolistic firm.**

(a) Find production level x_0 at which average costs are minimized.

(b) The demand function for the firm being $p(x) = 100 - x$, find the price that maximizes the profit of the firm.

Remark: justify your answers.

1 point

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a) Integrating $C'(x)$, and using the given fixed costs, we have that the total cost function is

$$C(x) = 4x^2 + 30x + 100.$$

The average cost function would then be

$$MC(x) = \frac{C(x)}{x} = 4x + 30 + \frac{100}{x}$$

So that $AC'(x) = 4 - 100/x^2$. It follows that the only critical point is $x_0 = 5$. Now we notice that $AC(x)$ is convex, since $AC''(x) = 200/x^3 > 0$ when $x > 0$, the relevant domain.

It follows that the critical point is the unique global minimum of the average cost function.

b) First, the profit function is

$$P(x) = (100 - x)x - (4x^2 + 30x + 100) = -5x^2 + 70x - 100.$$

If we set $P'(x) - 10x + 70$ to zero, we get that $x = 7$ is the only critical point. Now $P(x)$ is strictly concave, since $P''(x) = -10 < 0$, so that the critical point indeed maximizes benefits.

It follows that the price which maximizes profits is $p(7) = 100 - 7 = 93$.

4. **Given** $f(x) = xe^{1-x}$**,**

(a) Find the intervals of concavity and convexity, as well as the inflexion points of $f(x)$.

(b) Find the tangent line and Taylor's second degree polynomial at an inflexion point, and represent the graph of f around that point.

1 point

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a) Let's find the first and second derivatives of f:.

$$\begin{aligned} f'(x) &= e^{1-x} - xe^{1-x} = (1-x)e^{1-x} \\ f''(x) &= -e^{1-x} - (1-x)e^{1-x} = (x-2)e^{1-x} \end{aligned}$$

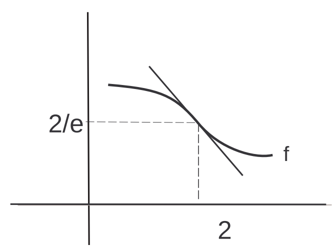
It follows that f is concave when $x \le 2$, since $f''(x) \le 0$ there. On the other hand, f is convex if $x \ge 2$, since $f''(x) \ge 0$ there. Hence $x = 2$ is the unique inflexion point of f.

b) The equation of the tangent line of f at $x_0 = 2$ is $y - f(2) = f'(2)(x-2)$. Since $f(2) = \frac{2}{e}$, $f'(2) = \frac{-1}{e}$, the tangent line equation is given by

$$y - \frac{2}{e} = -\frac{1}{e}(x-2).$$

Finally, since $f''(2) = 0$, Taylor's second degree polynomial has the same equation as the tangent line at that point.

A sketch of f around $(2, 2/e)$ is:



5. **Let A be the bounded set between curve $y = xe^{2x}$ and the lines $y = x$ and $x = 2$.**

(a) Graph the set A and find its maximals, minimals, maximum and minimum, if they exist.

(b) Find the area of A.

Hint: the Pareto order is given by: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

1 point

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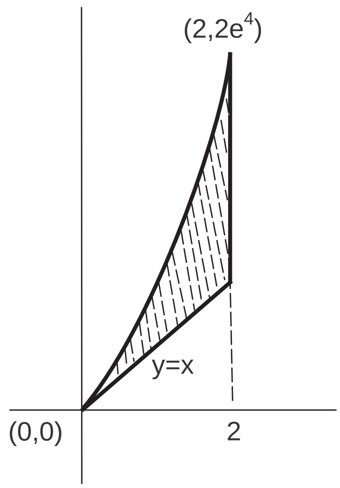
a) Since $e^{2x} > 1$ if $x > 0$ and $e^{2x} < 1$ if $x < 0$, we get that $xe^{2x} > x$ if $x \neq 0$.

Hence, the only bounded set determined by the above functions is contained in the first quadrant.

Its graph is:



Obviously,

$$\begin{aligned} maximum(A) &= maximals(A) = \{(2, 2e^4)\} \\ minimum(A) &= minimals(A)\} = \{(0,0)\} \end{aligned}$$

b) The area of A is:

$$\int_0^2 (xe^{2x} - x)dx$$

In order to solve this integral, we first find the primitive of xe^{2x}, by parts:

$$\begin{aligned} \int xe^{2x} &= x\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \\ &= x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} \end{aligned}$$

It follows that total area is

$$\begin{aligned} \int_0^2 (xe^{2x} - x)dx &= \left[x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} - \frac{x^2}{2} \right]_0^2 \\ &= 2\frac{e^4}{2} - \frac{e^4}{4} - \frac{2^2}{2} + \frac{1}{4} \\ &= \frac{3e^4}{4} - \frac{7}{4} \end{aligned}$$

6. **Given function**

$$f(x) = \frac{\ln(1+x)}{1+x},$$

(a) Find the primitive $F(x)$ of $f(x)$ such that $F(0) = 3$. Find the intervals at which F is increasing, decreasing, and find its minima and maxima on the interval $(-1, 1]$

(b) Find the Taylor polynomial of second degree of F at $a = 0$, and use it to approximate the value of $F(0,1)$.

Hint: for part b) you might not need to find $F(x)$.

1 point

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a) The primitive is immediate: $F(x) = \frac{1}{2}\ln^2(1+x) + C$. Since we should have $F(0) = 3$, it follows that $F(x) = \frac{1}{2}\ln^2(1+x) + 3$

Hence we have:

i) $F(x)$ is decreasing on $(-1, 0)$, since at that interval $F'(x) = f(x) < 0$.

ii) $F(x)$ is increasing on $(0, 1]$, since on that interval $F'(x) = f(x) > 0$.

It follows that $F(x)$ reaches a global minimum at $x = 0$. (where in fact $F'(x) = 0$).

On the other hand, $F(x)$ does not have a maximum at $x = -1$, where it is not defined (and is asymptotic there). It reaches a local maximum at $x = 1$.

b) Taylor's second degree polynomial for F at $a = 0$ is:

$$P(x) = F(0) + F'(0)x + \frac{1}{2}F''(0)x^2$$

He have

$$F''(x) = f'(x) = \frac{1 - \ln(1+x)}{(1+x)^2}$$

Since $F(0) = 3, F'(0) = f(0) = 0$, and $F''(0) = 1$, we have that

$$P(x) = 3 + \frac{1}{2}x^2.$$

Hence $F(0,1) \approx P(0,1) = 3 + \frac{1}{2}(0,01) = 3,005$