Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Final Exam Mathematics I** **June 24th 2011**

Duration of the exam: 2 hours.

LAST NAME:		**FIRST NAME:**
DNI:	**Degree:**	**Group:**

1. Consider the function $f(x) = \dfrac{\ln x}{x}$. We ask:

a) Find the domain, the intervals in which f is increasing/decreasing as well as all local and global maxima and minima, of f.

b) Find the asymptotes, the range (image) and represent the function graphically.

1 point

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a) The domain of the function is the interval $(0, \infty)$.

Since the derivative of the function is:

$f\,'(x) = \dfrac{1 - \ln x}{x^2}$, the sign of the derivative coincides withe that of $1 - \ln x$, so it follows that:

1) f increasing in the interval $(0, e]$, as $1 - \ln x > 0 \Longleftrightarrow \ln x < 1 \Longleftrightarrow x < e$; and

2) f decreasing in the interval $[e, \infty)$, as $1 - \ln x < 0 \Longleftrightarrow \ln x > 1 \Longleftrightarrow x > e$.

So it follows that the function attains a global and local maximum in e of value $f(e) = \dfrac{1}{e}$.

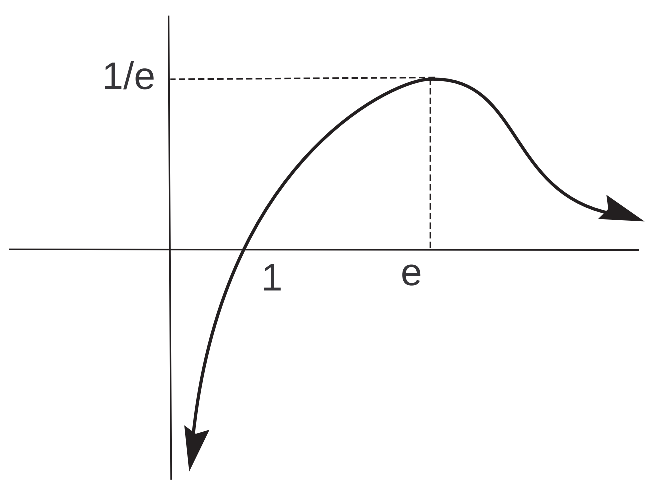
b) Concerning the asymptotes, as the function is continuous in its domain it may only have asymptotes at 0 and ∞.

$\lim\limits_{x \longrightarrow 0^+} f(x) = \dfrac{-\infty}{0^+} = -\infty$, so f has a vertical asymptote at $x = 0$; and

$\lim\limits_{x \longrightarrow \infty} f(x) = \dfrac{\infty}{\infty}$ =(by L'Hopital)= $\lim\limits_{x \longrightarrow \infty} \dfrac{1/x}{1} = 0$, so f has a horizontal asymptote, $y = 0$.

The function is continuous in its domain, increasing to the left of e and decreasing to the right and has a vertical asymptote at 0^+, the image is the interval $(-\infty, f(e)] = (-\infty, \dfrac{1}{e}]$.

Hence, the graph of f would look approximately like this:



2. Let $y = f(x)$ the function implicitly defined by the equation $xy^3 + 2x^2y = 3$, around the point $x = 1, y = 1$. We ask:

a) Find the tangent line of f in the point $x = 1, y = 1$.

b) Approximate, using the equation of the tangent line found before, the values of $f(0'9)$ y $f(1'2)$.

1 point

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a) First, we take the derivative of the equation defining the function to find:
$y^3 + 3xy^2y' + 4xy + 2x^2y' = 0$.
Substituting in this equation the points: $x = 1, y = 1$:
$1 + 3y' + 4 + 2y' = 0$, from this we conclude that $y' = -1$.
From this, the equation of the tangent line to the function f in the point $x = 1, y = 1$ will be:
$y = 1 + (-1)(x - 1)$ or in other words, $y = -x + 2$.

b) Obviously, the approximate values of the function in these points close to $x = 1$, are:
$f(0'9) \approx -0'9 + 2 = 1'1$
$f(1'2) \approx -1'2 + 2 = 0'8$

3. A company sells x units of a product at unit price $p(x) = 100 - 0,1x$.
Its production costs are $C(x) = 30x + C_0$, C_0 being fixed costs.

a) If the company can produce at most 300 units of this product in a certain time period, which unit price would maximize total profit?

b) Suppose that in the previous period the company produced 200 units and that it may increase its production, decrease it or keep it constant.
The government believes that this company is polluting and thus decided to impose a tax of 30 euros for each additional unit produced. For the same reason the government will pay a subsidy of 30 euros for every unit the company produces less.
What should the company do if it seeks to maximize its profits?
Hint: Check if the profit function is concave and if the marginal profit at $x = 200$ is 30.
1 point

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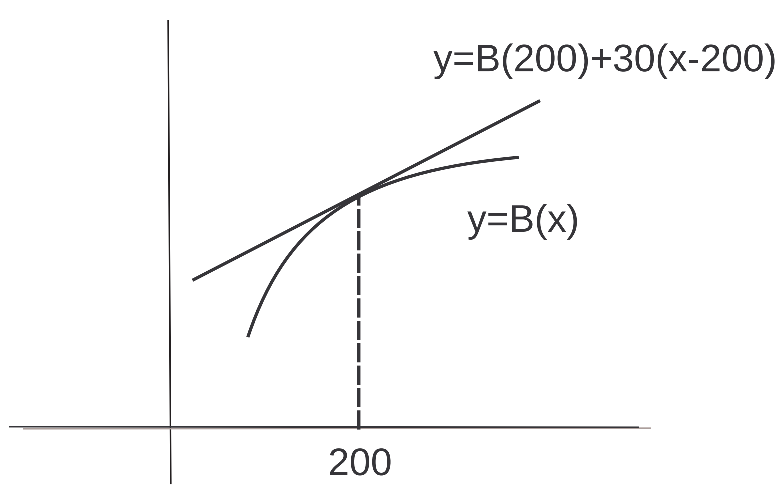
a) The profit function is: $B(x) = (100 - 0,1x)x - 30x - C_0 = -0,1x^2 + 70x - C_0$.
This function is increasing over its entire domain $[0, 300]$,
as its derivative is $B'(x) = -0,2x + 70 > 0$.
Hence, the profit is maximized choosing $x = 300$ and the unit price associated with this production level is $p(300) = 100 - 30 = 70$.

b) Given that the profit function is concave, as $B"(x) = -0,2 < 0$, the tangent line of said function in the point $x = 200$, lies above the graph of the profit function. Hence we obtain the following inequality:
$B(x) < B(200) + 30(x - 200)$ if $x \neq 200$; in other words, calling $\Delta x = x - 200$:
1) if $x > 200$, $B(200 + \Delta x) - 30\Delta x < B(200)$, so the company should not increase production. If it does its profit will be less than if it produced 200 units as the 30 euros per unit tax has to be taken into account.
2) if $x < 200$, $B(200+\Delta x)+30(-\Delta x) < B(200)$, so the company should not decrease its production. If it reduces it, its profit are less than if it produces 200 units despite the subsidy of 30 euros.
Hence, the company will maximize its profit by maintaining the previous level. The situation becomes clear from the following graph:



4. **Let the function** $f(x) = \begin{cases} x^3 + x + 3 & if \quad x < 0 \\ -2x^3 - x + 2 & if \quad 0 \leq x \end{cases}$ **, and consider the function** $f : [a, b] \to \mathbb{R}$, **where** $a < b$ **are integers. We ask:**

   a) Determine the interval(s) $[a, b]$ of length 1 that contain root(s) of f.

   b) Determine if in the following intervals the hypothesis and/or the thesis (or conclusion) of Bolzano's Theorem hold: $[-1, 1], [-3, -1], [1, 3]$.
   Hint: Draw the function and remember that the set of integers is: $... - 2, -1, 0, 1, 2, ...$
   1 point

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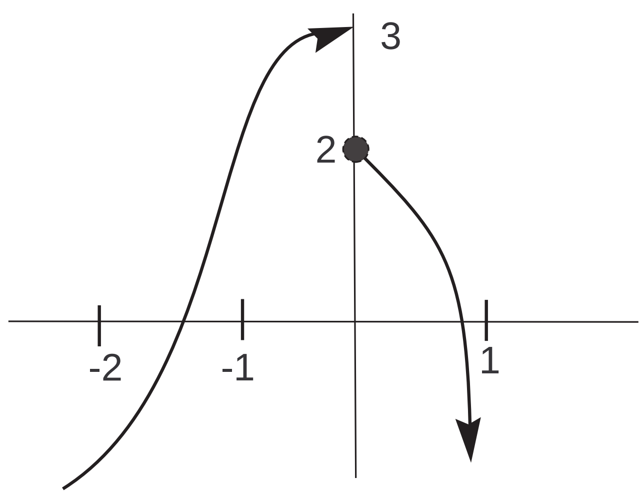
a) The function is only discontinuous at 0 from the left.
It is increasing in the interval $(-\infty, 0)$ and also satisfies: $f(-2) < 0,\ f(-1) > 0$.
Hence, the only negative root of f must be in the interval $[-2, -1]$.
On the other hand, it is decreasing in the interval $[0, \infty)$ and satisfies:
$f(0) > 0,\ f(1) < 0$.
Hence, the only positive root of f must be in the interval $[0, 1]$.
So, the intervals of length 1 containing the roots are: $[-2, -1]$ and $[0, 1]$.

b) In the first case the hypothesis is not satisfied, as f is not continuous in the said interval. Nevertheless, the thesis is satisfied, as there is a root in the interval.
In the second case, the hypothesis is satisfied, as the function is continuous and the corner points have different signs. Hence, the thesis is also saitisfied.
In the third case the hypothesis is not satisfied as the two corner points have the same sign. Neither is the thesis satisfied as there is no root in this interval.
This graph will help understand the situation:



5. **Consider the set A bordered by the function $f(x) = 0,5x^2 - 3,5x + 7$, the tangent line of f in the point $x = 2$, and the vertical axis. We ask:**

a) Draw the set A and find the maximals, minimals, maximum and minimum of A, if they exist.

b) Calculate the area of A.

Hint: the Pareto ordering is given by: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

1 point

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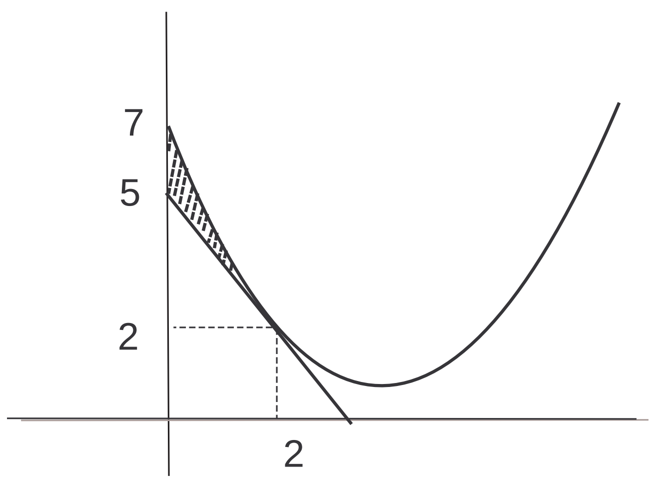
a) First, we find the equation of the tangent line.

As $f(2) = 2 - 7 + 7 = 2$, $f\,'(x) = x - 3,5 \Longrightarrow f\,'(2) = -1,5$, it follows that the tangent line is given by: $y = g(x) = 2 - 1,5(x-2) = 5 - 1,5x$.

Then, we find the interception points of the function and the tangent line with the vertical axis: $f(0) = 7, g(0) = 5$.

Finally, note that the function is convex, as $f"(x) = 1 > 0$, we find that the tangent line is always below the function.

So, the set A looks approximately like this:



We have then that:

Maximals(A)= parabol from $x = 0$ to $x = 2$

Minimals(A)= tangent line from $x = 0$ to $x = 2$.

Therefore A has neither maximum nor minimum.

(Remark: point $(2, 2)$ is both maximal and minimal!)

b) As the tangent line is always below the graph of the function, the area we are looking for is:

$$\int_0^2 [(0,5x^2 - 3,5x + 7) - (5 - 1,5x)]dx = \int_0^2 (0,5x^2 - 2x + 2)dx = \left[\frac{0,5x^3}{3} - x^2 + 2x\right]_0^2 =$$

$$= \frac{4}{3} - 4 + 4 = \frac{4}{3} \text{ area units.}$$

6. **Given the function** $f(x) = \dfrac{\ln x - 1}{x}$**, defined on** $x \in [1, \infty)$**, we ask:**

a) Find the intervals in which $F(x)$, the primitive of f with $F(1) = C$, is increasing or decreasing.

b) Determine C such that $F(x)$ has the horizontal axis as tangent line.

Hint: Draw the graph of $F(x)$ when this function has the horizontal axis as tangent line.

1 point

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a) From $F'(x) = \dfrac{\ln x - 1}{x}$, it follows that for every primitive F of f you find that:
$F(x)$ increasing $\Longleftrightarrow F'(x) > 0 \Longleftrightarrow \ln x - 1 > 0 \Longleftrightarrow x > e$
$F(x)$ decreasing $\Longleftrightarrow F'(x) < 0 \Longleftrightarrow \ln x - 1 < 0 \Longleftrightarrow 1 < x < e.$
On the other hand, as $\int \frac{\ln x - 1}{x} = \int \frac{\ln x}{x} - \int \frac{1}{x} = \frac{(\ln x)^2}{2} - \ln x$, it follows that $F(x) = \frac{(\ln x)^2}{2} - \ln x + C$.

b) From $F'(x) = f(x) = \dfrac{\ln x - 1}{x}$, it follows that:
$F'(x) = 0 \Longleftrightarrow \ln x = 1 \Longleftrightarrow x = e$; and now, as the horizontal line is tangent to this line $F(x)$, will satisfy:
$F(e) = 0 \Longleftrightarrow \dfrac{(\ln e)^2}{2} - \ln e + C = 0 \Longleftrightarrow \frac{1}{2} - 1 + C = 0 \Longleftrightarrow C = \frac{1}{2}.$
So the graph of F looks like this: