Universidad Carlos III de Madrid

Question	1	2	3	4	5	6	total
Grade							

Economics Department **Final Exam, Mathematics I** **June 28, 2010**

Total time length: 2 hours.

LAST NAMES:	**FIRST NAME:**
DNI: **Title:**	**Group:**

1. Let $g(x) = |x-1| - 1$. We ask:

a) Find the domain, the image (range) and the intervals at which g is increasing/decreasing.

b) Let h be defined the same as g, but restricted to the interval $(-\infty, 1]$.
Find the inverse of h, as well as the domain and image of that inverse function.
Hint: Consider first the function $f(x) = |x|$. Starting from it, represent $g(x), h(x)$ and $h^{-1}(x)$.
1 point

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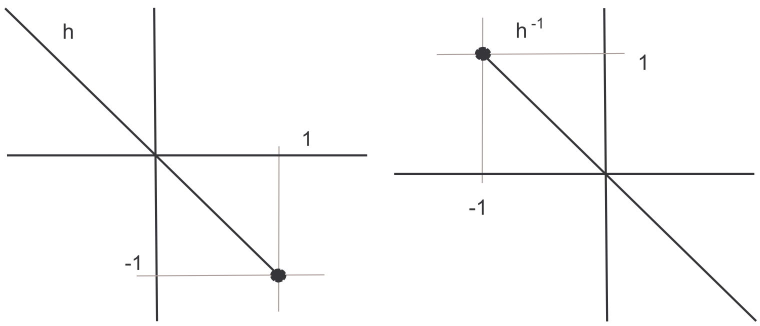
a) Function $g(x)$ can be difined by sections as:

$$g(x) = \begin{cases} -(x-1) - 1 = -x & if\ x \leq 1 \\ (x-1) - 1 = x - 2 & if\ \ x \geq 1 \end{cases}$$

It follows that the domain is the real lilne, is decreasing in the interval $(-\infty, 1]$ and increasing in the interval $[1, \infty)$. Since $g(1) = -1$ and has limit ∞ (resp. $-\infty$) at ∞ (resp. $-\infty$), its range is the interval $[-1, \infty)$.

b) Since $g : (-\infty, 1] \to [-1, \infty)$ is onto, we infer that $g^{-1}(x) = -x$, its domain is $[-1, \infty)$ and its image is $(-\infty, 1]$.
The graphs of g and g^{-1} are:



2. Let a be a real number, and consider the following function defined by sections:

$$f(x) = \begin{cases} \dfrac{x}{1+x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \ln(x^2+1) & \text{if } x > 0 \end{cases}$$

a) Analyze, acording to the values of a, the derivability of $f(x)$ at $x = 0$.

b) Analyze the existence of asymptotes of $f(x)$.

1 point

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a) First of all, let us see if the function is continuous at $x = 0$.

Notice that

$\lim\limits_{x \longrightarrow 0^-} f(x) = 0$, $f(0) = a$, $\lim\limits_{x \longrightarrow 0^+} f(x) = 0$, so that f is continuous at $x = 0$ when $a = 0$.

Assuming now that $f(x)$ is continuous at $x = 0$, $f(x)$ is derivable at $x = 0$ when

$\lim\limits_{x \longrightarrow 0^-} f\,'(x) = \lim\limits_{x \longrightarrow 0^+} f\,'(x)$. Since

$\lim\limits_{x \longrightarrow 0^-} f\,'(x) = \lim\limits_{x \longrightarrow 0^-} \dfrac{1+x^2-2x^2}{(1+x^2)^2} = 1$; and

$\lim\limits_{x \longrightarrow 0^+} f\,'(x) = \lim\limits_{x \longrightarrow 0^+} \dfrac{2x}{x^2+1} = 0$;

we conclude that there is no a for which f is derivable at $x = 0$.

b) Since the function is continuous everywhere, it does not have a vertical asymptote.

Regarding possible asymptotes at $-\infty$, since

$\lim\limits_{x \longrightarrow -\infty} f(x) = \lim\limits_{x \longrightarrow -\infty} \dfrac{x}{1+x^2} = 0$,

$f(x)$ has a horizontal asymptote (hence not oblique) of $y = 0$ at $-\infty$.

Regarding possible asymptotes at ∞, since

$\lim\limits_{x \longrightarrow \infty} f(x) = \lim\limits_{x \longrightarrow \infty} \ln(x^2+1) = \infty$; and

$\lim\limits_{x \longrightarrow \infty} \dfrac{f(x)}{x} = \lim\limits_{x \longrightarrow \infty} \dfrac{\ln(x^2+1)}{x} = \frac{\infty}{\infty}$ =(L'Hopital)= $\lim\limits_{x \longrightarrow \infty} \dfrac{2x}{x^2+1} = 0$,

$f(x)$ has no horizontal or oblique asymptote at ∞.

3. Let $f(x) = 2ae^x - e^{2ax}$ **, for** $0 \neq a \neq \frac{1}{2}$**. We ask:**

a) Find the Taylor polynomial of degree 2 at $x_0 = 0$.

b) Determine the values of a for which the function f reaches a local maximum or minimum at the point $x_0 = 0$.

1 point

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a) We have that $f(x) = 2ae^x - e^{2ax}, f(0) = 2a - 1$;
$f\,'(x) = 2ae^x - 2ae^{2ax},\ f\,'(0) = 0$
and $f\,''(x) = 2ae^{ax} - 4a^2e^{2ax}, f''(0) = 2a - 4a^2$.
It follows that the Taylor polynomial of degree 2 at $x_0 = 0$ is $P_2(x) = \frac{1}{2}(2a - 4a^2)x^2 + 2a - 1$.

b) f reaches a local minimum at $x_0 = 0$ when $a(1 - 2a) > 0$
or, equivalently, when $0 < a < \frac{1}{2}$.
By the same token, f reaches a local maximum at $x_0 = 0$ when
$a(1 - 2a) < 0$ or, equivalently, when $a < 0$ or when $a > \frac{1}{2}$.

4. Let $C(x) = C_0 + x + 0,01x^2$ and $p(x) = a - \dfrac{x}{50}$ be the cost and demand functions, respectively, of a monopolistic firm. We ask:

a) Find a and C_0 for which the medium cost is minimum at production $x = 100$.

b) Find a and C_0 for which benefits are maximum at production $x = 100$.

1 point

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a) The medium cost function is $\frac{C(x)}{x} = \frac{C_o}{x} + 1 + 0'01x$. In order to get that $x = 100$ minimizes that function, it should be the case that $\left(\frac{C(x)}{x}\right)' = \frac{-C_o}{x^2} + 0'01 = 0$. Equivalently, $x^2 = \frac{C_0}{0'01} = 100C_0$, or $C_0 = 100$, for any value of a. In addition, $x = 100$ is a point of global minimum, since $C(x)/x$ is convex

b) The benefits function is $B(x) = p(x).x - C(x) = (a - \dfrac{x}{50})x - (C_0 + x + 0,01x^2)$. In order to get maximum benefits, we need
$B'(x) = a - 0'04x - 1 - 0,02x = 0$. So, it is required that $a = 1 + 0'06x$. Since $x = 100$ should be the solution, we need $a = 7$, for any value of C_0. In addition, $x = 100$ is a point of global maximum, since $B(x)$ is concave.

5. Consider the function $f(x) = \frac{1}{x}$, and $g(x)$, the tangent line to the graph of f at the point $x = 1$. We ask:

a) Draw the closed recint A limited by the functions f, g, the horizontal axis, and the line $x = 3$. Find the maximals and minimals, and the maximum and minimum of A, if they exist.

b) Find the area of A.

1 point

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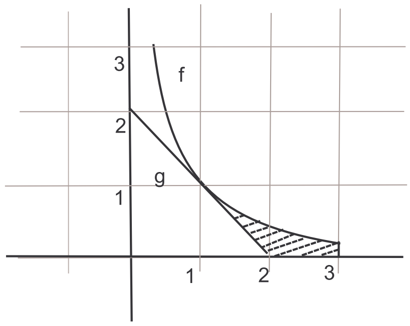
a) The graph of $f(x) = \frac{1}{x}$ is an equilateral hiperbola. The tangent lline at $x = 1$ is the function $g(x) = 2 - x$.
Such tangent line is below the hiperbola (since the funcion is convex)
and cuts the horizontal axis at $x = 2$.
Hence, re recint can be drawn like this:



Obvioulsly, $\max(A), \min(A)$ do not exist , since maximals(A)= $\{(x, y) : y = \frac{1}{x}, 1 \le x \le 3\}$ minimals(A)= $\{(x, y) : y = 2 - x, 1 \le x \le 2\}$.

b) The set A can be understood as the union of two other sets:
1) set A_1 bounded by functions f, g and the line $x = 2$.
The value of its area is:
$\int_1^2 (\frac{1}{x} - (2 - x))dx = \left[\ln x - 2x + \frac{1}{2}x^2\right]_1^2 = \ln 2 - 4 + 2 - (0 - 2 + \frac{1}{2}) =$
$= \ln 2 - \frac{1}{2}$
2) set A_2 bounded by functions f, the horizontal axis and the lines $x = 2, x = 3$. The value of its area is:
$\int_2^3 \frac{1}{x}dx = [\ln x]_2^3 = \ln 3 - \ln 2$
Hence, the area os A is:
$\text{area}(A) = \ln 2 - \frac{1}{2} + \ln 3 - \ln 2 = \ln 3 - \frac{1}{2}$

6. Let $f(x) = a\ln(ax)$, where $a > 0$. We ask:

a) Find the indefinite integral $F(x) = \int f(x)dx$.

b) Determine the value of a so that the tangent line to $f(x)$ at $x = 2$ goes through the point $(0, 0)$.

Remark: you can solve b) without solving a).

1 point

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a) Using the fact that $\ln(ax) = \ln a + \ln x$, we have that
$\int a\ln(ax)dx = (a\ln a)x + a\int \ln x dx = (a\ln a)x + a(x\ln x - x) + C = ax\ln(a + x) - ax + C$

b) Since $f\,'(x) = \frac{a}{x}$, $f\,'(2) = \frac{a}{2}$, we conclude that the tangent line to f at $x = 2$ is:
$y - a\ln(2a) = \frac{a}{2}(x - 2)$.
Since that line goes through the origin, it satisfies:
$-a\ln(2a) = \frac{a}{2}(-2) \Longrightarrow \ln(2a) = 1 \Longrightarrow a = \frac{e}{2}$.
Hence, the equation of the tangent line is:
$y - \frac{e}{2}\ln(e) = \frac{e}{4}(x - 2)$, or $y - \frac{e}{2} = \frac{e}{4}(x - 2)$.