Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **January 10th 2020**

Exam time: 2 hours.

LAST NAME: **FIRST NAME:**

ID: **DEGREE:** **GROUP:**

(1) **Consider the function $f(x) = x^4 e^x$. Then:**

(a) find the domain and the asymptotes of $f(x)$.

(b) find the intervals where $f(x)$ increases and decreases, its global maximum and minimum and range (or image).

(c) draw the graph of the function.

0.3 points part a); 0.5 points part b); 0.2 points part c)

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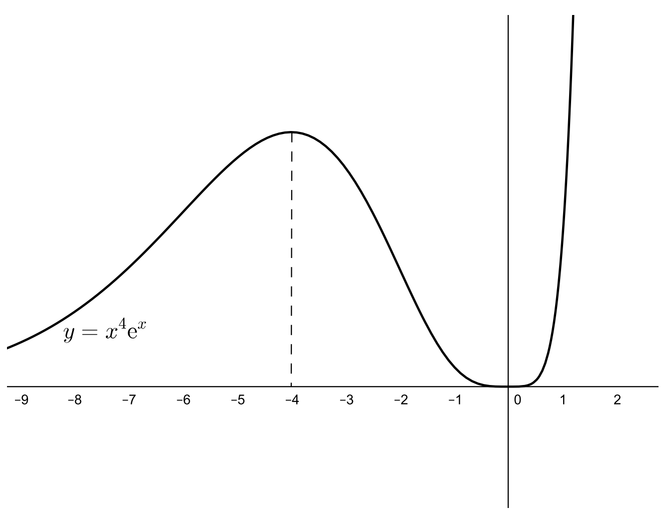
a) the domain of the function is $\mathbb{R}$.
Since f is continuous on its domain, we only need to study its asymptotes at ∞ and $-\infty$:
i) $\lim\limits_{x\longrightarrow-\infty} f(x) = \lim\limits_{x\longrightarrow-\infty} \dfrac{x^4}{e^{-x}} = \frac{\infty}{\infty} =$ [applying L'Hopital's Rule four times] =
$= \lim\limits_{x\longrightarrow-\infty} \dfrac{24}{e^{-x}} = \dfrac{24}{\infty} = 0.$
Therefore $f(x)$ has a horizontal asymptote $y = 0$ at $-\infty$.
ii) $\lim\limits_{x\longrightarrow\infty} \dfrac{f(x)}{x} = \lim\limits_{x\longrightarrow\infty} x^3 e^x = \infty$, so f has no horizontal neither oblique asymptote at ∞.

b) As $f'(x) = e^x(x^4 + 4x^3)$, we find that $f'(-4) = f'(0) = 0$ and we can deduce:
f is increasing $\Longleftrightarrow f'(x) > 0 \Longleftrightarrow x^4 + 4x^3 = x^3(x+4) > 0$; then f is increasing on $(-\infty, -4]$ and $[0, \infty)$. Analogously, f is decreasing on $[-4, 0]$.
Interpreting the monotonicity of $f(x)$ it is deduced that -4 is a local maximum point and 0 is a local minimum point.
Since $\lim\limits_{x\longrightarrow\infty} f(x) = \infty$, there is no global maximum.
In addition, as $f(0) = 0$ and $f(x) \geq 0$, it is deduced that 0 is a strict (unique) global minimum point.
Finally, as $f(0) = 0, f(x) \geq 0$ and $\lim\limits_{x\longrightarrow\infty} f(x) = \infty$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $[0, \infty)$.

c) The graph of f will have an appearance approximately, similar to:



(2) **Given the implicit function** $y = f(x)$, **defined by the equation** $2x + 2y - e^{y-x} = -1$ **in a neighbourhood of the point**$x = 0, y = 0$, **it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 0$.

(b) sketch the graph of the function f near the point $x = 0, y = 0$. Sketch the graph of f^{-1} near the point $x = 0, y = 0$, using the tangent line to the graph of f^{-1} at that point $x = 0, y = 0$. Justify the convexity or concavity of f and f^{-1}.

Hint for part b: consider the symmetry between f and f^{-1}.

0.5 points part a); 0.5 points part b)

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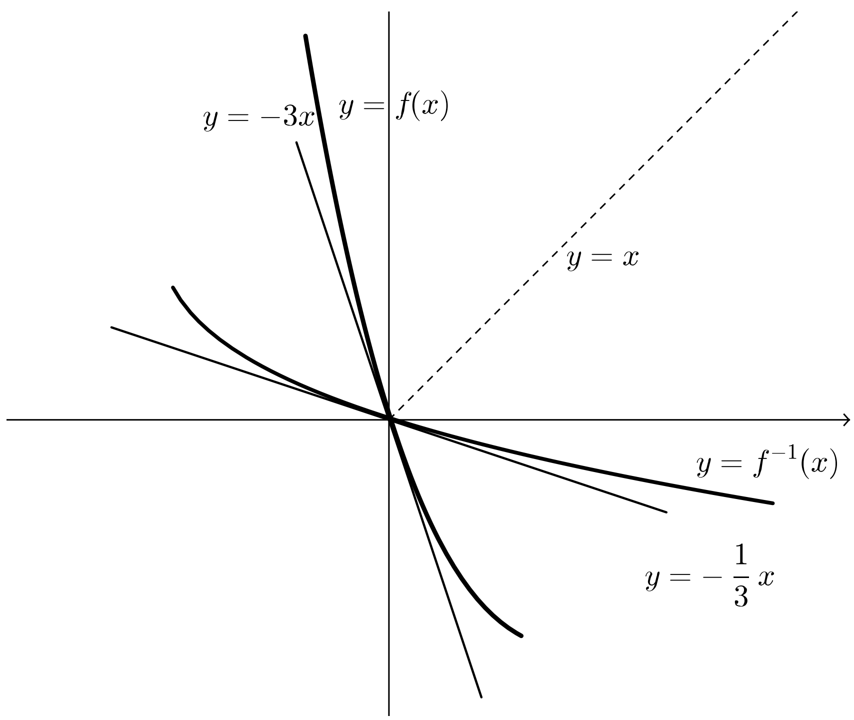
a) First of all, we calculate the first-order derivative of the function: $2 + 2y' - e^{y-x}(y' - 1) = 0$ evaluating at $x = 0,\ y(0) = 0$ we obtain $y'(0) = f'(0) = -3$.
Then the equation of the tangent line is: $y = P_1(x) = 0 - 3(x - 0) = -3x$.
Secondly, we calculate the second-order derivative of the function:
$2y'' - e^{y-x}[(y' - 1)^2 + y''] = 0$ evaluating at $x = 0, y(0) = 0, y'(0) = -3$ we obtain
$y''(0) = f''(0) = 16$.
Therefore the second-order Taylor Polynomial is: $y = P_2(x) = 0 - 3x + 8x^2$.

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be as you can see on the figure at the bottom.
Furthermore, as $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(0)} = -\frac{1}{3}$.
then the tangent line to f^{-1} at $x = 1$ will be: $y = -\frac{1}{3}x$.
Since f is convex and decreasing in a neighbourhood of $x = 0$, by symmetry we know that f^{-1} will be convex and decreasing as well, near the point $x = 0$.
And, you can see the graph of f^{-1} near the point $x = 0$ on the figure at the bottom.



(3) **Let** $C(x) = 75 + 80x - x^2$ **be the cost function and** $p(x) = 200 - 3x$ the inverse demand function of a monopolistic firm, being $0 \leq x \leq 40$ **the number of units produced of certain goods. Then:**

(a) find the price p^* and the quantity x^* in order to obtain the maximum profit.

(b) If the government increases the cost through a tax T euros per produced unit, find the new quantity $x^*(T)$ and the new price $p^*(T)$ that maximize the company profit.
Compare the results between both cases.
0.5 points part a); 0.5 points part b)

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a) First of all, we calculate the profit function.
$B(x) = (200 - 3x)x - (75 + 80x - x^2) = -2x^2 + 120x - 75$
Secondly, we calculate the first and second order derivative of B :
$B'(x) = -4x + 120; B''(x) = -4 < 0$
we see that B has a critical point at $x^* = \dfrac{120}{4} = 30$ and, as B is a concave function, this critical point is the unique global maximizer.
Finally, $p^* = p(30) = 200 - 90 = 110$ euros.

b) As the new cost function is $C(x) = 75 + (80 + T)x - x^2$, then the related profit function is $B(x) = -2x^2 + (120 - T)x - 75$.
Since $B'(x) = -4x + 120 - T; B''(x) = -4 < 0$,
we can see that B has an unique critical point at $x^*(T) = \dfrac{120 - T}{4} = 30 - \dfrac{T}{4}$.
Furthermore, B is a concave function so the critical point is the unique global maximizer.
Finally, $p^*(T) = 200 - 3(30 - \dfrac{T}{4}) = 110 + 3\dfrac{T}{4}$ euros.
It is appreciated that the produced quantity has decreased and the related price has increased.

(4) **Let** $f(x) = \begin{cases} x^2+6x+a & si\ x \leq 1 \\ bx+2 & si\ \ x > 1 \end{cases}$ **be a piece-wise defined function in the interval**$[0,2]$**.**

Then:

(a) Calculate a and b such that $f(x)$ satisfies the hypothesis of the Mean Value Theorem (or Lagrange's Theorem).

(b) For the values $a = 2, b = 8$, are the hypothesis of the theorem satisfied? There is any value of c such that the thesis (or conclusion) of Lagrange's Theorem is satisfied, although the hypothesis is not satisfied?

Hint for parts a) and b): state Lagrange's Theorem. *Hint for part b):* If there is more than one value of c, you do not need to calculate all of them.

0.5 points part a); 0.5 points part b)

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a) The hypothesis of the theorem are satisfied when f is continuous in $[0,2]$ and derivable in $(0,2)$. Then, we need to impose the continuity and differentiability of f at $x = 1$.

Since $\lim_{x \longrightarrow 1^-} f(x) = 7 + a = f(1)$, $\lim_{x \longrightarrow 1^+} f(x) = b + 2$

we can assume that the function will be continuous at the point if: $7 + a = b + 2$.

Moreover, supposing that the function is continuous at $x = 1$, will be differentiable at the point if: $8 = f'_-(1) = f'_+(1) = b$.

Then the function will be continuous and differentiable at $x = 1$ when:

$7 + a = b + 2, 8 = b \Longleftrightarrow a = 3, b = 8$.

Then Lagrange's Theorem hypothesis is satisfied when $a = 3, b = 8$.

b) For the values $a = 2, b = 8$ the hypothesis of theorem is not satisfied because f is not differenciable at the point $x = 1$, since it is not continuous. However, the thesis or conclusion of the theorem can still be satisfied, this is:

(*) there is $c \in (0,2) : f(2) - f(0) = f'(c)(2 - 0)$.

Bearing in mind that $a = 2, b = 8 \Longrightarrow f(2) = 18,\ f(0) = 2$,

(*) means that there is $c \in (0,2) : 18 - 2 = 2f'(c)$ and this is equal to: $f'(c) = 8$.

Therefore, every $c \in (1,2)$ satisfies the thesis or conclusion of the theorem.

(5) **Given the functions** $f, g : [0,1] \longrightarrow \mathbb{R}$ **defined by:** $f(x) = -\ln(1+x), g(x) = 5 - e^x$, **Then:**

(a) draw approximately the set $A = \{(x,y) : 0 \leq x \leq 1, f(x) \leq y \leq g(x)\}$ and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for part a: Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b)

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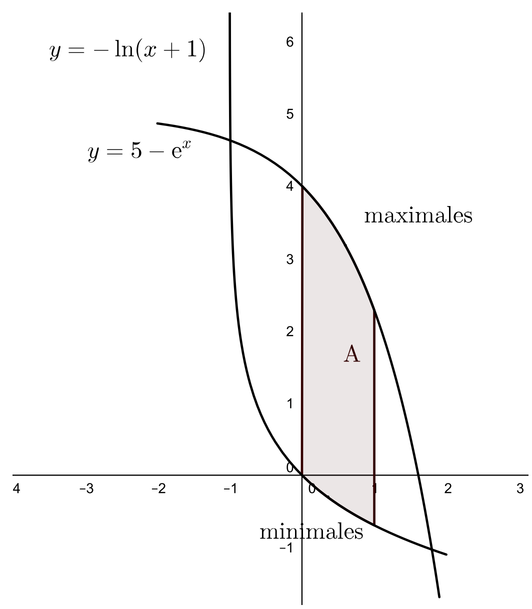
a) Both $f\ (x)$ and $g(x)$ are decreasing on their domains, as they have negative derived functions. Moreover, for any x between 0 and 1, it is satisfied:

i) $f(x) \leq 0$ always, since $f(0) = 0$; and

ii) $0 < g(x)$ always, since $g(1) > 0$

So, the drawing of A will be, approximately, this way:



with this graph, the Pareto order describes the set in the following way:

there is no maximum, maximals$(A) = \{(x, g(x)) : 0 \leq x \leq 1\}$,

there is no minimum, minimals$(A)\} = \{(x, f(x) : 0 \leq x \leq 1\}$.

b) First of all, we calculate the primitive function of $g(x)$, integrating by parts:

$\int 1 \cdot (-\ln(x+1)) = \int u'v = uv - \int uv' = x(-\ln(x+1)) - \int x\dfrac{(-1)}{x+1} =$

$= -x\ln(x+1) + \int \dfrac{x}{x+1} = -x\ln(x+1) + \int \dfrac{x+1}{x+1} - \int \dfrac{1}{x+1} =$

$= -x\ln(x+1) + x - \ln(x+1) = x - (x+1)\ln(x+1)$

Then applying Barrow's Rule we obtain:

$\int_0^1 (g(x) - f(x))dx = [5x - e^x - x + (x+1)\ln(x+1)]_0^1 = (5 - e - 1 + 2\ln(2)) - (-1) = 5 - e + 2\ln(2) =$

$5 - e + \ln(4)$ area units.

(6) **Given the function** $g(x) = \dfrac{10-4x}{2+x^3}$, **then:**

(a) prove that $\int_0^2 g(t)dt$ is a number between 2.2 and 7.

(b) Sketch approximately, the graph of the function $G(x) = \int_0^x g(t)dt$ defined on the interval $[0,2]$, obtaining firstly, its increasing and decreasing intervals, its global maximum and minimum points, its convex and concave intervals and its inflection points.

Hint for parts a) and b): prove that $g(x)$ is decreasing.

0.6 points part a); 0.4 points part b)

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a) $g(x)$ is increasing on $[0,2]$, since in the given interval the numerator is decreasing, the denominator increasing and both are positive. It is deduced:
$g(1) + g(2) < \int_0^2 g(t)dt < g(0) + g(1)$.
As, $g(0) = 5$, $g(1) = 2$, $g(2) = 0.2$ the statement is proved.

b) i) $G(x)$ is increasing on $[0,2]$, since its derivative is $g(x) > 0$.
ii) $G(x)$ is concave on $[0,2]$, as is derivative $g(x)$, is decreasing.
Therefore, its global minimum is attained at the point $x = 0$, and the maximum is attained at $x = 2$.
Moreover, $G(x)$ has no inflection points.
So, the drawing of $G(x)$ will be approximately, similar to: