Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **January 16th 2019**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = \sqrt{x^2 - x}$. Then:**

(a) find its domain, its asymptotes, the intervals where $f(x)$ increases and decreases, its global maximum and minimum and range (or image).

(b) study the convexity and concavity of the function and draw the graph of the function and its asymptotes.

0.6 points part a); 0.4 points part b)

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a) The domain of the given function is $\{x : x^2 - x = x(x-1) \geq 0\} = (-\infty, 0] \cup [1, \infty)$.
Since f is continuous on its domain, which is the union of closed intervals, we only need to study its asymptotes at ∞ and $-\infty$:

i) $\displaystyle\lim_{x \longrightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \longrightarrow \pm\infty} \frac{\sqrt{x^2 - x}}{\pm\sqrt{x^2}} = \pm \lim_{x \longrightarrow \pm\infty} \sqrt{1 - \tfrac{1}{x}} = \pm 1.$

ii) $\displaystyle\lim_{x \longrightarrow \pm\infty} [f(x) - \pm(x)] = \lim_{x \longrightarrow \pm\infty} [\sqrt{x^2 - x} - \sqrt{x^2}] =$[rationalizing:]

$\displaystyle= \lim_{x \longrightarrow \pm\infty} [\sqrt{x^2 - x} - \sqrt{x^2}][\sqrt{x^2 - x} + \sqrt{x^2}]/[\sqrt{x^2 - x} + \sqrt{x^2}] =$

$\displaystyle= \lim_{x \longrightarrow \pm\infty} \frac{x^2 - x - x^2}{\sqrt{x^2 - x} + \sqrt{x^2}} = - \lim_{x \longrightarrow \pm\infty} \frac{x}{\sqrt{x^2 - x} + \sqrt{x^2}} =$

[dividing the numerator and denominator by $x = \sqrt{x^2}$]

$\displaystyle= - \lim_{x \longrightarrow \pm\infty} \frac{\pm 1}{\sqrt{1 - 1/x} + 1} = \mp \tfrac{1}{2};$

iii) therefore f has an oblique asymptote $y = \pm\, x \mp \frac{1}{2}$ at $\pm\infty$.

In addition, as $f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x}}$, we can deduce that:

f is increasing if $\iff f'(x) > 0 \iff 2x - 1 > 0$; then f is increasing on $[1, \infty)$.

Analogously, f is decreasing on $(-\infty, 0]$.

Since $f(0) = f(1) = 0$ and $f(x) \geq 0$, it is deduced that 0 and 1 are the global minimum points, and because $\lim_{x \longrightarrow \infty} f(x) = \infty$, it is known that there is no global maximum.

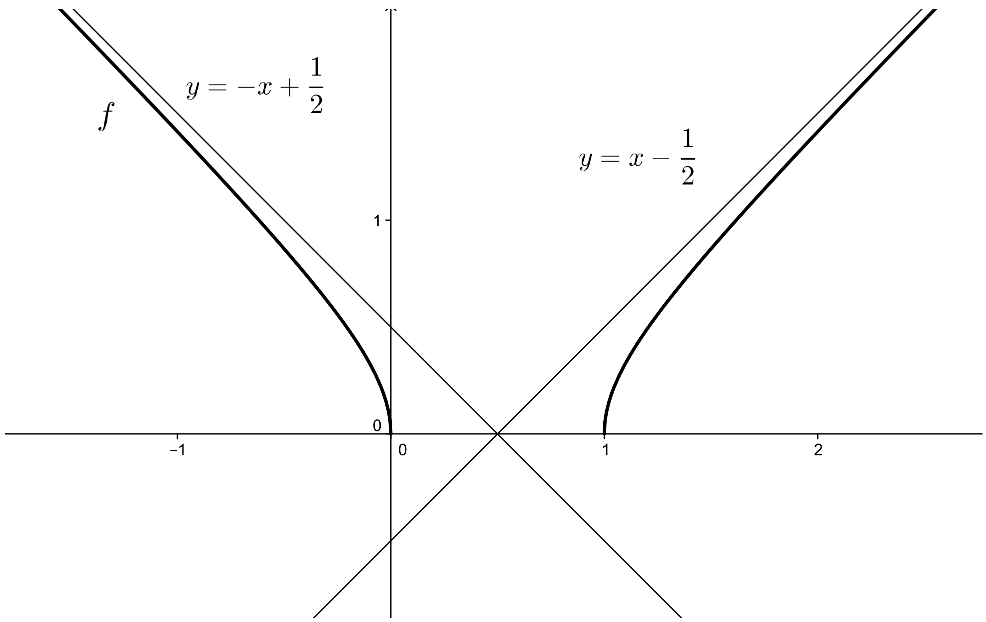
Finally, as $f(1) = 0$, $f(x) \geq 0$ and $\lim_{x \longrightarrow \infty} f(x) = \infty$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $[0, \infty)$.

b) Since $f''(x) = \left(\dfrac{2x - 1}{2\sqrt{x^2 - x}}\right)' = \dfrac{4\sqrt{x^2 - x} - (2x-1)^2/\sqrt{x^2 - x}}{4(x^2 - x)} < 0$, and in the domain of the function this inequality is equivalent to:

$4\sqrt{x^2 - x} < (2x-1)^2/\sqrt{x^2 - x} \iff 4(x^2 - x) < (2x - 1)^2 \iff 0 < 1.$

Therefore, the function is concave on $(-\infty, 0]$ and on $[1, \infty)$,

and the asymptotes at $\pm\infty$ lay above the graph of the function. The graph of f will have an appearance approximately, similar to:



$y = -x + \frac{1}{2}$
f
$y = x - \frac{1}{2}$
1
0
-1
0
1
2

(2) **Given the implicit function $y = f(x)$, defined by the equation $ye^{-x} - y^3 - 3x = 0$ in a neighbourhood of the point $x = 0, y = 1$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 0$.

(b) sketch the graph of the function f near the point $x = 0, y = 1$. Sketch the graph of f^{-1} near the point $x = 1, y = 0$. using the tangent line to the graph of f^{-1} at that point. Justify the convexity or concavity of f and f^{-1}.

Hint for part b: consider the symmetry between f and f^{-1}.

0.5 points part a); 0.5 points part b)

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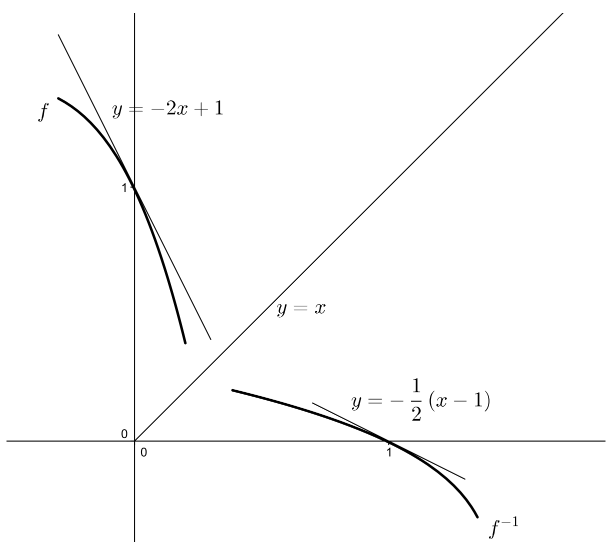
a) First of all, we calculate the first-order derivative of the function:
$y'e^{-x} - ye^{-x} - 3y^2y' - 3 = (y' - y)e^{-x} - 3[y^2y' + 1] = 0$
evaluating at $x = 0,\ y(0) = 1$ we obtain $y'(0) = f\,'(0) = -2$.
Then the equation of the tangent line is: $y = P_1(x) = 1 - 2(x - 0) = 1 - 2x$.

Secondly, we calculate the second-order derivative of the function:
$(y'' - y')e^{-x} - (y' - y)e^{-x} - 3[2y(y')^2 + y^2y''] = 0$
evaluating at $y(0) = 1, y'(0) = -2$ we obtain $y''(0) = f''\ (0) = -\frac{19}{2}$.
Therefore the second-order Taylor Polynomial is: $y = P_2(x) = 1 - 2x - \frac{19}{4}x^2$.

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be as you can see on the left-hand side of the figure at the bottom.
Furthermore, as $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = -\frac{1}{2}$, the tangent line of f^{-1} at $x = 1$ is: $y - 0 = (-\frac{1}{2})(x - 1)$.
Since f is concave and decreasing in a neighbourhood of $x = 0$, by symmetry we know that f^{-1} will be concave and decreasing as well, near the point $x = 1$.
And, you can see the graph of f^{-1} near the point $x = 1$ on the right-hand side of the figure at the bottom.



(3) **Let $C(x) = 4000 - 40x + 0.02x^2$ be the cost function and $p(x) = 50 - 0.01x$ the inverse demand function of a monopolistic firm, being $x \geqslant 0$ the number of units produced of certain goods. Then:**

(a) find the price p^* and the quantity x^* in order to obtain the maximum profit.

(b) find the marginal revenue at the point x^*. Justify that the obtained value approximates, the variation of the revenue function in each case, when we sell one unit more or one unit less than x^*. Describe, for both cases, if the approximation is rounded up or down for the variation of revenue. *Hint*: Remember that the revenue function is $I(x) = p(x) \cdot x$ and $I'(x)$ is the marginal revenue.
0.5 points part a); 0.5 points part b)

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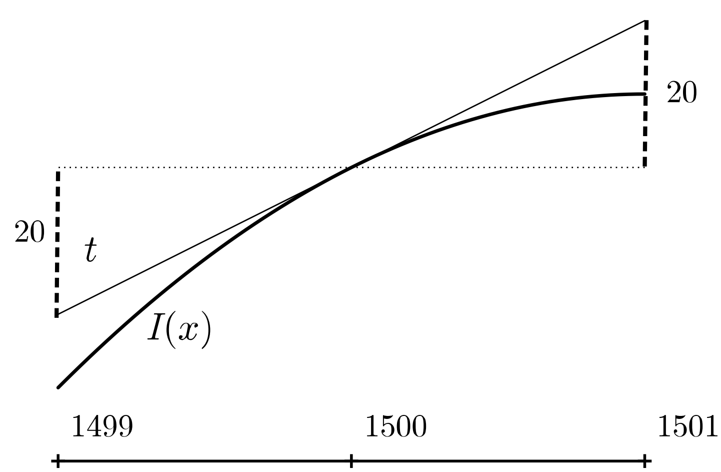
a) First of all, we calculate the profit function.
$B(x) = (50 - 0.01x)x - (4000 - 40x + 0.02x^2) = -0.03x^2 + 90x - 4000$
Secondly we calculate the first and second order derivative of B :
$B'(x) = -0.06x + 90$; $B''(x) = -0.06 < 0$
we see that B has a critical point at $x^* = \dfrac{90}{0.06} = 1500$ and, as B is a concave function, this critical point is the unique global maximizer.
Finally, $p^* = p(1500) = 50 - 0.01 \cdot 1500 = 35$

b) Since the revenue function is $I(x) = (50 - 0.01x)x$, its marginal function is $I'(x) = 50 - 0.02x$. Thus $I'(1500) = 20$.
Applying the mean value theorem, two numbers exist $c^- \in (1499, 1500)$ and $c^+ \in (1500, 1501)$ such that:
i) $I(1500) - I(1499) = I'(c^-) \cdot (1500 - 1499) = I'(c^-) \approx I'(1500) = 20$.
ii) $I(1501) - I(1500) = I'(c^+) \cdot (1501 - 1500) = I'(c^+) \approx I'(1500) = 20$. So both variations can be approximated by the marginal function at the point. Furthermore, because $I''(x) < 0$, the revenue function is concave, then the marginal revenue function $I'(x)$ is decreasing and we deduce:
i) the income reduction when we produce one less unit will be:
$I(1500) - I(1499) = I'(c^-) > I'(1500) = 20$, that is greater than 20 monetary units and the approximation is rounded down; and
ii) the income increase when we produce one more unit will be:
$I(1501) - I(1500) = I'(c^+) < I'(1500) = 20$, that is less than 20 monetary units and the approximation is rounded up.
The following drawing describes the situation, where $t(x)$ is the tangent line of $I(x)$ at $x = 1500$:



(4) **Let** $f(x) = \begin{cases} 1 + ax^2 & si\ x < -1 \\ abx & si\ \ x \geqslant -1 \end{cases}$ **be a piece-wise defined function in the interval** $[-3, 1]$**.**

Then:

(a) Calculate a and b such that $f(x)$ satisfies the hypothesis of the Mean Value Theorem (or Lagrange's Theorem).

(b) For the values $a = -1, b = -1$, find the intermediate numbers c such that the thesis (or conclusion) of Lagrange's Theorem is satisfied, although the hypothesis is not satisfied.

Hint for parts a) and b): state Lagrange's Theorem.

0.5 points part a); 0.5 points part b)

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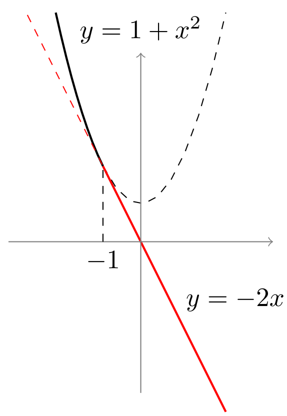
a) The hypothesis of the theorem are satisfied when f is continuous in $[-3, 1]$ and derivable in $(-3, 1)$. Then, we need to impose the continuity and differentiability of f at $x = -1$.
Since $\lim_{x \longrightarrow -1^-} f(x) = 1 + a$, $f(-1) = \lim_{x \longrightarrow -1^+} f(x) = -ab$
we can assume that the function will be continuous at the point if: $1 + a = -ab$.
Moreover, supposing that the function is continuous at $x = -1$,
will be differentiable at the point if: $-2a = f'(-1^-) = f'(-1^+) = ab$.
Then the function will be continuous and differentiable at $x = -1$ when:
$1 + a = -ab$, $-2a = ab \Longrightarrow$
i) if $a = 0$, the first equation is not satisfied; and
ii) if $a \neq 0$, from the second equation we deduce that $b = -2$ and $a = 1$ from the first.
Then Lagrange's Theorem hypothesis is satisfied when $a = 1, b = -2$. The graph of that function is:



b) If the thesis of Lagrange's Theorem is satisfied, we'll have:
(*) there is a $c \in (-3, 1) : f(1) - f(-3) = f'(c)(1 - (-3))$.
Bearing in mind that $a = -1, b = -1 \Longrightarrow f(1) = 1,\ f(-3) = -8$
therefore, (*) means that $1 - (-8) = f'(c)4$. that is: $f'(c) = 9/4$.
i) If $-3 < x < -1$, then $f'(x) = -2x$, and
$f'(c) = -2c = 9/4 \Longleftrightarrow c = -9/8$.
ii) If $-1 < x < 1$, then $f'(x) = 1 \neq 9/4$;
and $c = -9/8$ is the only number that satisfies Lagrange's Theorem.

(5) **Let** $f, g : [0,3] \longrightarrow \mathbb{R}$ **be the functions defined by:** $f(x) = -e^x, g(x) = \ln(4-x)$ **Then:**

(a) draw approximately the set $A = \{(x,y) : 0 \leq x \leq 3, f(x) \leq y \leq g(x)\}$ and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for part a: Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b)

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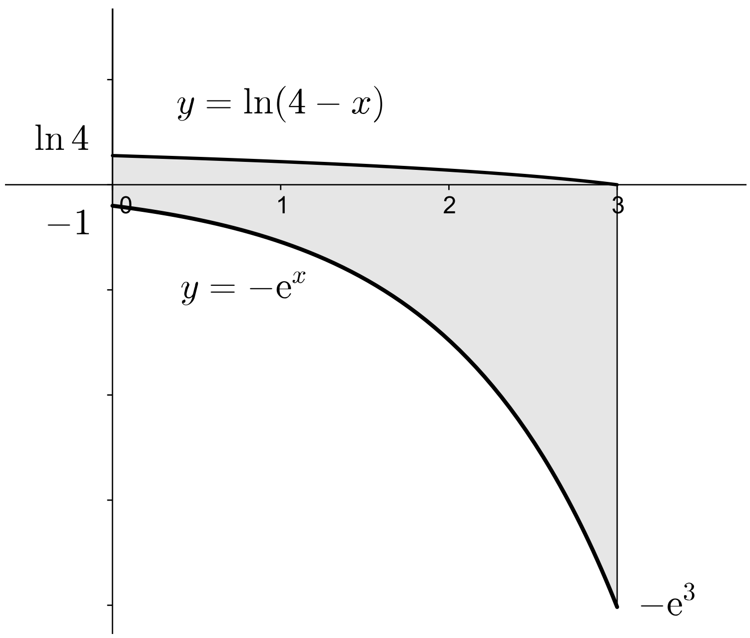
a) Both $f(x)$ and $g(x)$ are decreasing on their domains, as they have negative derived functions. So, the drawing of A will be, approximately, this way:



with this graph, the Pareto order describes the set in the following way:
there is no maximum, maximals$(A) = \{(x, g(x)) : 0 \leq x \leq 3\}$.
There is no minimum minimals$(A) = \{(x, f(x)) : 0 \leq x \leq 3\}$.

b) First of all, we calculate the primitive function of $g(x)$, integrating by parts:

$\int 1 \cdot \ln(4-x)dx = \left(\int h'g = hg - \int hg'\right) = x\ln(4-x) - \int x\dfrac{(-1)}{4-x}dx = x\ln(4-x) - \int \dfrac{4-x-4}{4-x}dx =$

$= x\ln(4-x) - \int \dfrac{4-x}{4-x}dx - 4\int \dfrac{(-1)}{4-x}dx = x\ln(4-x) - x - 4\ln(4-x) = (x-4)\ln(4-x) - x$

Then applying Barrow's Rule we obtain:

$\int_0^3 (g(x) - f(x))dx = [(x-4)\ln(4-x) - x + e^x]_0^3 = (-3+e^3) - (-4\ln 4 + 1) =$

$= 4(\ln 4 - 1) + e^3$ area units.

(6) **Given the function** $g(x) = \dfrac{x}{10+x^6}$**, then:**

(a) Sketch approximately, the graph of the function $G_1(x) = \int_{-8}^{x} g(t)dt$ defined on the interval $[-8, 9]$, obtaining firstly its increasing and decreasing intervals and global maximum and minimum.

(b) Let now G_2 be a function such that $G_2'(x) = g(x)$ and $G_2(0) = 0$.
Find the second-order Taylor Polynomial of $G_2(x)$ centered at $a = 0$.
Use the polynomial to calculate approximately the area bounded by the horizontal axis, the graph of $g(x)$ and the vertical lines $x = 0$, $x = 0.1$.
Hint for parts a) and b): don't try to find a formula for $G_1(x)$ or $G_2(x)$.
0.6 points part a); 0.4 points part b)

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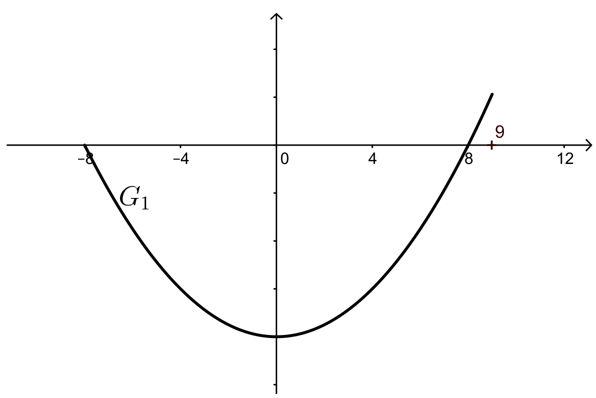
a) Since $G_1'(x) = \dfrac{x}{10+x^6}$, we know that:
i) $G_1(x)$ is decreasing on $[-8, 0]$, because it has negative derivative.
ii) $G_1(x)$ is increasing on $[0, 9]$, because it has positive derivative.
Thus, its global minimum is attained at $x = 0$.
To find its maximum, as $g(x)$ is an odd function, we know that $\int_{-8}^{8} g(t)dt = 0$. Therefore $G_1(-8) = 0 = G_1(8) < G_1(9)$, because $G_1(x)$ is increasing on $[0, 9]$.
Thus, its global maximum is attained at $x = 9$.
So, the drawing of $G_1(x)$ will be approximately, similar to:



b) The asked area is $\int_0^{0.1} g(t)dt = G_2(0.1)$.
Considering $G_2'(x) = g(x)$, we have $G_2'(0) = 0$.
And considering $G_2''(x) = g'(x) = (\dfrac{x}{10+x^6})' = \dfrac{10+x^6-6x^6}{(10+x^6)^2}$, we have $G_2''(0) = \frac{1}{10}$.
Therefore, second-order Taylor Polynomial of $G_2(x)$ is: $P_2(x) = \frac{1}{20}x^2$.
Then the approximate value of the area will be: $P_2(0.1) = \frac{1}{20}0.1^2 = \frac{1}{2000}$.