Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **January 22nd 2018**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = \ln(x) + \dfrac{1}{(x-2)}$**. Then:**

(a) draw the graph of the function, obtaining firstly its domain, its vertical asymptotes, the intervals where $f(x)$ increases and decreases, local and global extrema and range (or image).

(b) consider the new function $f_1(x) = f(x)$ (only defined on the interval $[4, \infty)$. Draw the graphs of $f_1(x)$, its inverse and the first quadrant bisector line, and discuss the relative position between them.

Hint: from the value of $f_1(4)$ and $f_1{}'(x)$ you should be able to find the relative position between $f_1(x)$ and the first quadrant bisector. Notice that $\ln 4 < 2$. Don't try to find the analytical expression of $f_1^{-1}(x)$.

0.6 points part a); 0.4 points part b)

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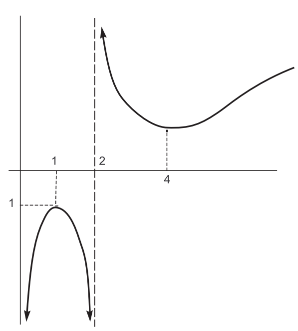
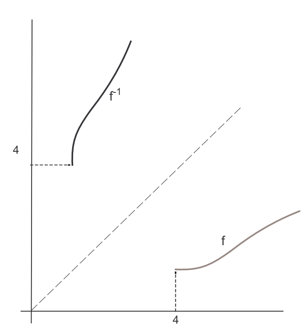
a) The domain of the function is $\{x : x > 0, x \neq 2\} = (0,2) \cup (2, \infty)$. Since the function is continuous in its domain, we only need to study its vertical asymptotes at $0^+, 2^-$ and 2^+ :
$\lim\limits_{x \longrightarrow 0^+} f(x) = \ln(0^+) - \frac{1}{2} = -\infty - \frac{1}{2} = -\infty$; so, the function has a vertical asymptote at $x = 0^+$. $\lim\limits_{x \longrightarrow 2^-} f(x) = \ln(2) + \frac{1}{0^-} = \ln 2 - \infty = -\infty$; , a vertical asymptote at $x = 2^-$. $\lim\limits_{x \longrightarrow 2^+} f(x) = \ln(2) + \frac{1}{0^+} = \ln 2 + \infty = \infty$; and a vertical asymptote at $x = 2^+$.
On the other hand, as $f\ '(x) = \dfrac{1}{x} - \dfrac{1}{(x-2)^2}$, we can deduce that: f is increasing if $\Longleftrightarrow f\ '(x) = \dfrac{1}{x} - \dfrac{1}{(x-2)^2} > 0 \Longleftrightarrow x^2 - 5x + 4 = (x-1)(x-4) > 0, x \neq 2$; then f is increasing on $(0, 1]$ and on $[4, \infty)$ but decreasing on $[1, 2)$ and on $(2, 4]$
therefore we can say that f attains a local maximum at $x = 1$ and a local minimum at $x = 4$.
Moreover, as $\lim\limits_{x \longrightarrow 2^-} f(x) = -\infty$, $\lim\limits_{x \longrightarrow 2^+} f(x) = \infty$, the function doesn't attain any global extrema.
Finally, because the function is continuous on the intervals $(0, 2)$ and on $(2, \infty)$, by the intermediate values theorem the image will be $(-\infty,\ f(1)] \cup [f(4), \infty) = (-\infty, -1] \cup [\frac{1}{2} + \ln 4, \infty)$.
Conclusion: the graph of f will have an appearance, approximately, similar to the first drawing (A):

b) As $f_1(4) = \frac{1}{2} + \ln 4 < 4$, $f_1{}'(x) = \dfrac{1}{x} - \dfrac{1}{(x-2)^2} < \dfrac{1}{x} < 1$, the graph of f_1 is underneath of the first quadrant bisector $y = x$, since f_1 starts in a smaller value at $x = 4$ and is increasing slower than the straight line $y = x$. By symmetry the graph of $f_1^{-1}(x)$ is above the bisector.
Therefore, the relative position of the three graphs can be approximately sketched as in the second figure (B).



(A)



(B)

(2) **Let $y = f(x)$ be an implicit function defined by the equation $xe^{-y} + y^2 = 1$ in a neighborhood of the point $x = 1, y = 0$. Then:**

(a) find the tangent line and the Taylor polynomial of degree 2 of the function centered at $a = 1$.

(b) Sketch the graph of f near the point $x = 1$. Use the tangent line to obtain an approximation of the values of $f(1.1)$ and $f(0.9)$.
Can you show if any of these approximations are rounded up or rounded down?
1 point

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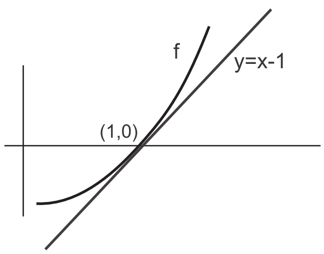
a) First of all, we compute the first derivative of the function:
$e^{-y} - xy'e^{-y} + 2yy' = e^{-y}(1 - xy') + 2yy' = 0$ substituting $x = 1,\ y(1) = 0$ in the previous equation, you obtain: $y'(1) = f'(1) = 1$.
So, the equation of the tangent line will be: $y = P_1(x) = 0 + 1(x - 1)$, that is, $y = x - 1$.
Analogously, we compute the second derivative of the function:
$-e^{-y}y'(1 - xy') + e^{-y}(-y' - xy") + 2y'y' + 2yy'' = 0$
substituting $y(1) = 0, y'(1) = 1$ in this last equation, you obtain: $y''(1) = f''(1) = 1$
So, the equation of the Taylor polynomial will be: $y = P_2(x) = x - 1 + \frac{1}{2}(x - 1)^2$.

b) Using second order Taylor polynomial, the graph of f will be close to the figure:



On the other hand, the first order approximations are:
$f(1.1) \approx P_1(1.1) = 0.1$; $f(0.9) \approx P_1(0.9) = -0.1$
Because $f''(1) > 0$, the function is convex in a neighbourhood of $x = 1$, so the approximations of the values of f obtained by the tangent line are rounded down in both cases.

(3) **Let $C(x) = C_0 + 9x + 2x^2$ be the cost function and $p(x) = 81 - 4x$ be the inverse demand function of a monopolistic firm, being $x \geqslant 0$ the number of units produced of certain goods and $a > 0$. Then:**

(a) find the fixed costs C_0 such that 200 euros is the maximum profit.

(b) find the fixed costs C_0 such that the minimum average cost is attained at $x = 3$.
What is that minimum value of the average cost?
0.5 points part a); 0.5 points part b)

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a) First of all, we calculate the profit function:
$B(x) = (81 - 4x)x - (C_0 + 9x + 2x^2) = -6x^2 + 72x - C_0$
Secondly, we calculate the first and second order derivative of $B:$
$B'(x) = -12x + 72; B''(x) = -12 < 0$
so we see that B has an unique critical point when $x = \dfrac{72}{12} = 6$ and, as B is a concave function, this critical point is the unique global maximizer.
Then $B(6) = 6(-36 + 72) - C_0 = 216 - C_0 = 200 \Longrightarrow C_0 = 16.$

b) The average cost function is $C_m(x) = \dfrac{C(x)}{x} = \dfrac{C_0}{x} + 9 + 2x.$
If we calculate the first and second order derivatives of this function:
$C'_m(x) = \dfrac{-C_0}{x^2} + 2; C''_m(x) = 2\dfrac{C_0}{x^3} > 0$
we observe that $x = 3$ is the only critical point of the function $C_m(x)$ when $C_0 = 18.$
and, as this function is convex, that critical point is the only global minimizer.
Therefore, the production that minimizes the average cost is: $x = 3.$
Finally, substituting in the average cost function, the minimum average cost will be:
$C_m(3) = \dfrac{18}{3} + 9 + 2.3 = 6 + 9 + 6 = 21.$

(4) **Let** $f(x) = \begin{cases} 1 + \dfrac{a}{(x-4)} & if \ x < 2 \\ a + \dfrac{b}{\sqrt{x+2}} & if \ \ x \geqslant 2 \end{cases}$ **be a piecewise defined function on the interval** $[1, 7]$**.**

Then:

(a) calculate a and b so that $f(x)$ satisfies the hypothesis (or initial conditions) of Lagrange's Theorem (or Mean Value Theorem) on that interval.

(b) for these a, b values find the intermediate value or values c, such that the thesis (or conclusion) of this theorem is satisfied.

Hint for part a): state the Mean Value Theorem. *Hint for part b)*: use 2.6 as an approximation of $6/\sqrt{5}$, and only consider the case $1 < c \leqslant 2$.

0.6 points part a); 0.4 points part b)

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a) We need to set the continuity and derivability at $x = 2$.
For that reason, as $\lim\limits_{x \longrightarrow 2^-} f(x) = 1 - a/2$, $f(2) = \lim\limits_{x \longrightarrow 2^+} f(x) = a + b/2$
it can be deduced that the function will be continuous at that point when:
$1 - a/2 = a + b/2 \Longleftrightarrow 3a + b = 2$.
On the other hand, supposing the function is continuous at $x = 2$, it will have a derivative at that point if:
$-a/4 = f\ '_-(2) = f\ '_+(2) = (-b/2)(1/8) \Longleftrightarrow 4a = b$.
So the function will be continuous and derivable at $x = 2$ when $a = 2/7, b = 8/7$.

b) By the mean value theorem we know that:
(*) There exists $c \in (1, 7) : f(7) - f(1) = f\ '(c)(7-1)$.
Taking into account that $a = 2/7, b = 8/7 \Longrightarrow$
$f(1) = 1 - a/3 = 19/21$, $f(7) = a + b/3 = 2/7 + 8/21 = 14/21$
we notice that (*) is equivalent to $14/21 - 19/21 = -5/21 = 6f\ '(c)$. In other words: $f\ '(c) = (-5/7)(1/18)$.
In the first case, if $1 < x \leq 2$, we have $f\ '(x) = -2/7(x-4)^2$, then
$f\ '(x) = -2/7(x-4)^2 = (-5/7)(1/18) \Longleftrightarrow 36/5 = (x-4)^2 \Longleftrightarrow \pm 2,6 = x - 4$
so it is not possible that $2,6 = x - 4$, since $x = 6,6$ y $x \leq 2$,
so it must be $-2,6 = x - 4$ and the only possible value is $x = 1,4$.
In other words, there exists $c \in (1, 2) : f(7) - f(1) = f\ '(c)(7-1)$.
The second case, when $2 \leq x < 7$; then as $f\ '(x) = (-b/2)(x+2)^{-3/2}$,
and $f\ '(x) = (-5/7)(1/18)$ it is equivalent to:
$(-4/7)(x+2)^{-3/2} = (-5/7)(1/18) \Longleftrightarrow 72/5 = (x+2)^{3/2} \Longleftrightarrow$
$\Longleftrightarrow (x+2)^3 = 14,4^2 \in (14^2, 15^2) = (196, 225)$
Since $h(x) = (x+2)^3$ satisfies $h(3) = 125, h(5) = 343$,
exists $c \in (3, 5) : (c+2)^3 = 14,4^2$,
or equivalently $c \in (3, 5) : f(7) - f(1) = f\ '(c)(7-1)$.

(5) **Given the function** $f : [0,2] \longrightarrow \mathbb{R}$ **define by:** $f(x) = xe^{-x}$, **then:**

(a) draw approximately the set $A = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq f(x)\}$ and find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for a: Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b)

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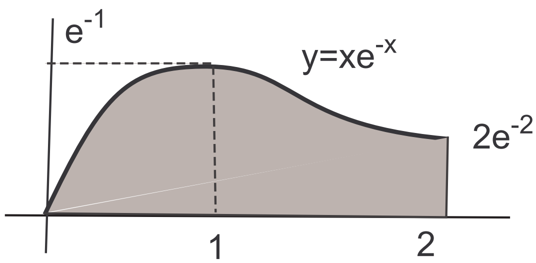
a) As $f\,'(x) = e^{-x}(1-x)$, it means that the function is increasing on the interval $[0,1]$ and decreasing on the interval $[1,2]$. So, the drawing of A will be, approximately, this way:



by this graph, the Pareto order describes the set in the following way: there is not a maximum of A, maximals$(A) = \{(x, f(x)) : 1 \leq x \leq 2\}$. minimum$(A)$ = minimals$(A)\} = \{(0,0)\}$.

b) First of all, we calculate the primitive function of $f(x)$, integrating by parts:

$\int xe^{-x} = \int fg' = fg - \int f\,'g = x(-e^{-x}) - \int 1(-e^{-x}) = x(-e^{-x}) + \int e^{-x} = (x+1)(-e^{-x})$

Then applying Barrow's Rule we obtain:

$\int_0^2 f(x)dx = [(x+1)(-e^{-x})]_0^2 = 3(-e^{-2}) - (-1) = 1 - 3e^{-2} =$ Area (A).

(6) **Given the function** $g(x) = x^2 - 4x + 3$**, then:**

(a) draw the region delimited by the graph of $g(x)$, the tangent line to the function at the point $x = 0$ and the horizontal axis.

(b) Let g a decreasing convex function on the interval $[0, 1]$ such that passes through the points $(0, 3)$ and $(1, 0)$. Calculate the best lower (or rounded down) and upper (or rounded up) approximation of the area of the region given in part a).

Hint for b: draw the region for both approximations when the tangent line crosses the horizontal axis at any possible point in the interval $(0, 1)$.

1 point

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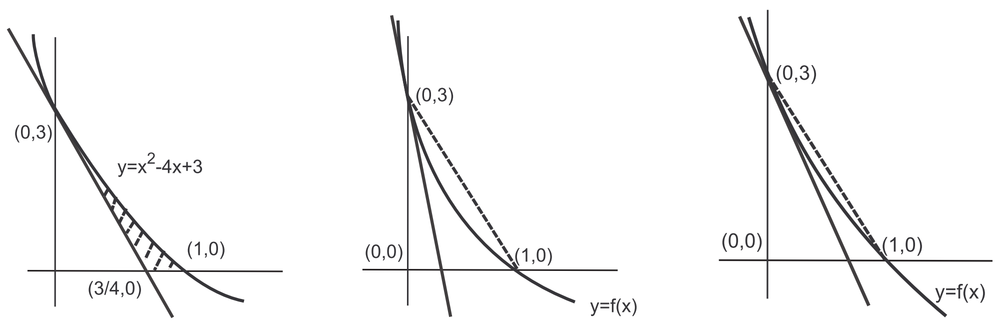
a) The equation of the tantgent line at $x = 0$ is: $y - g(0) = g'(0)(x - 0)$, that is, $y - 3 = (-4)x$, so will intercept the horizontal axis, when $y = 0$, at the point $x : 0 - 3 = -4x \iff x = 3/4$.
Moreover, the function $g(x) = (x-1)(x-3)$ intercepts the horizontal axis at the points $x = 1, x = 3$. Furthermore, the function g is decreasing on the interval $[0, 1]$ (since $g'(x) < 0$) and it is convex (since $g"(x) > 0$), then the graph of g is above the tangent line. See figure C.

b) the best rounded up approximation of the value of the area is $\frac{3}{2}$, since the region is always included in the triangle T_+ whose vertices are (0,0), (0,3) and (1,0), because of the convexity of g.
Analogously, the best rounded down approximation of value of the area is equal to 0, since the region could be included in the triangle T_- that is arbitrarily small whose vertices are: $(1 - \epsilon, 0), (1, 0)$ y $(0, 3)$. the figures D and E can help you to understand these situations:



(C) (D) (E)