<u>Universidad Carlos III de Madrid</u>

Exercise	1	2	3	4	5	6	Total
Points							

Department de Economics **Mathematicas I Final Exam** **January 20th 2017**

Exam time: 2 hours.

LAST NAME: **FIRST NAME:**

ID: **DEGREE:** **GROUP:**

(1) **Consider the function** $f(x) = \ln(|x| - 2) + 3$. **Then:**

(a) Draw the graph of the function, obtaining firstly its domain, symmetries and intersecting points with both axis, intervals where $f(x)$ increases and decreases, asymptotes and range of $f(x)$.

(b) Consider the new function $f_1(x) = f(x)$, defined only on the interval where that function is increasing. Draw the graphs of $f_1(x)$, its inverse and their corresponding tangent lines at the point $x = 3$, using the concavity and / or convexity of $f_1(x)$ and $f_1^{-1}(x)$.

Hint: you can find the analytical expression of $f_1^{-1}(x)$, but it is not necessary.

0.6 points part a); 0.4 points part b)

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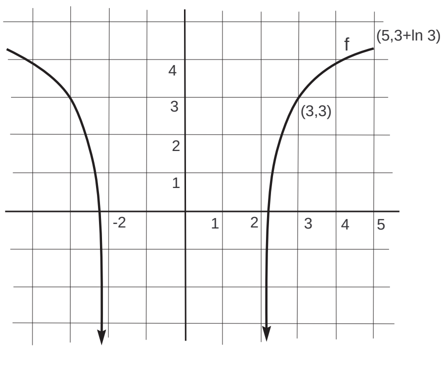
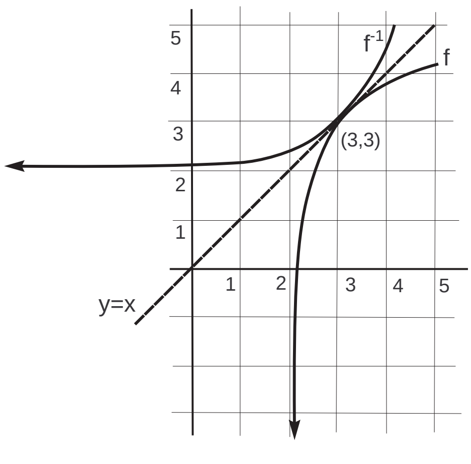
a) The domain of the function is $\{x : |x| > 2\} = (-\infty, -2) \cup (2, \infty)$.
On the other hand, the function is even, so its graph will be symmetric with respect to the vertical axis. For that reason, it is enough to study it on the interval $(2, \infty)$.
The graph has no intersection points with the vertical axis, because $x = 0$ is not in the domain of the function. Regarding the horizontal axis, if $x > 2$ its intersection point will be x: $\ln(x-2) = -3 \Longleftrightarrow x - 2 = e^{-3} \Longleftrightarrow x = 2 + e^{-3}$. And, by symmetry, if $x < -2$, the intersection point will be $x = -2 - e^{-3}$.
Moreover, as $f\,'(x) = \dfrac{1}{x-2}$ (if $x > 2$), you can deduce that f is increasing on $(2, \infty)$ and, by symmetry, decreasing on $(-\infty, -2)$.
As the function is continuous on its domain, it is only necessary to study the possible vertical asymptotes in 2^+ and in -2^- :
$\lim_{x \longrightarrow 2^+} f(x) = \ln(0^+) + 3 = -\infty + 3 = -\infty$; so the function has a vertical asymptote in $x = 2^+$ and, by symmetry, in $x = -2^-$. And, about the asymptotes in $\pm\infty$, as:
$\lim_{x \longrightarrow \infty} f(x) = \infty$; $\lim_{x \longrightarrow \infty} \dfrac{f(x)}{x} = 0$; so the function does not have any horizontal or oblique asymptotes in ∞ and, by symmetry, neither in $-\infty$.
For that reason, as $\lim_{x \longrightarrow 2^+} f(x) = -\infty$, $\lim_{x \longrightarrow \infty} f(x) = \infty$ and the function is continuous on the interval $(2, \infty)$, by the intermediate values theorem the image will be $(-\infty, \infty)$.
Conclusion: the graph of f will have an appearance, approximately, as the first drawing:





b) If $y = f_1^{-1}(x) > 2$, $f_1(y) = \ln(y-2) + 3 = x \Longleftrightarrow \ln(y-2) = x - 3 \Longleftrightarrow$

$\iff y - 2 = e^{x-3} \iff y = f_1^{-1}(x) = 2 + e^{x-3}$. But the problem can be solved without computing the analytical expression of the inverse function.
As $f(3) = \ln(3-2) + 3 = 3$, $f_1^{-1}(3) = 3$. Analogously, as
$f\,'(x) = \dfrac{1}{x-2} \Longrightarrow f\,'(3) = 1 \Longrightarrow (f_1^{-1})'(3) = 1$, so both functions have the same tangent line $y - 3 = x - 3$ (i.e., $y = x$) in $x = 3$.
Finally, as f_1 is a concave function, because $f"(x) = \dfrac{-1}{(x-2)^2} < 0$, and increasing, it can be deduced that $f_1^{-1}(x)$ is both convex and increasing.
For those reasons, the graph of f_1 will be drawn under its tangent line $y = x$, and the graph of f_1^{-1} will remain above the same line, and both graphs will intersect at the point $(3, 3)$.
The relative position of both graphs, with respect to their common tangent line at the point $x = 3$, will be, near the point $(3, 3)$, approximately, as the second drawing.

(2) **Let $y = f(x)$ be an implicit function defined by the equation $ye^x - y^2x + x = 1$, in a neighbourhood of the point $x = 0, y = 1$. Then:**

(a) Find the tangent line and the Taylor polynomial of degree 2 of the function centered at $a = 0$.

(b) Use the tangent line and the Taylor polynomial to obtain an approximation of the value of $f(0,1)$. Can you justify if any of these approximations are by default or by excess?

0.5 points part a); 0.5 points part b)

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a) First of all, we compute the first derivative of the function:
$ye^x + y'e^x - 2yy'x - y^2 + 1 = 0$
substituting $x = 0,\ y(0) = 1$ in the previous equation, you obtain:
$y'(0) = f\ '(0) = -1$.
Analogously, we compute the second derivative of the function:
$(y" + 2y' + y)e^x - 2(y')^2x - 2yy"x - 4yy' = 0$
substituting $x = 0$, $y(0) = 1, y'(0) = -1$ in this last equation, you obtain:
$y"(0) = f\ "(0) = -3$
So, the equation of the tangent line will be: $y = P_1(x) = 1 + (-1)(x - 0)$, i.e. $y = 1 - x$.
And the equation of the Taylor polynomial will be: $y = P_2(x) = 1 - x - \frac{3}{2}x^2$
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b) First order approximation: $f(0,1) \approx P_1(0,1) = 0.9$.
Second order approximation: $f(0,1) \approx P_2(0,1) = 0.885$.
As the function is concave near the point $x = 0$, because $f"(0) < 0$, the approximation by the tangent line is by excess. We cannot claim anything about the second order approximation.

(3) **Let $C(x) = 72 + 9x + 2x^2$ be the cost function and $p(x) = 81 - ax$ be the inverse demand function of a monopolistic firm, being $x \geqslant 0$ the number of units produced of certain good and $a > 0$. Then:**

(a) Find the production that maximizes the profit.

(b) Find the minimum average cost.

Hint for b: first of all, find the production that minimizes the average cost.

0.5 points part a); 0.5 points part b)

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a) First of all, we calculate the profit function:
$B(x) = (81 - ax)x - (72 + 9x + 2x^2) = (-a-2)x^2 + 72x - 72$
Secondly, we compute the first and second order derivative of B :
$B'(x) = (-a-2)2x + 72; B"(x) = (-a-2)2 < 0$
so we see that B has an unique critical point when $x = \dfrac{36}{(a+2)}$ and, as B is a concave function, this critical point is the unique global maximizer.

b) The average cost function is $C_m(x) = \dfrac{C(x)}{x} = \dfrac{72}{x} + 9 + 2x$.
If we compute the first and second order derivatives of this function:
$C'_m(x) = \dfrac{-72}{x^2} + 2; C"_m(x) = 2\dfrac{72}{x^3} > 0$
we observe that $x = 6$ is the only critical point of the function $C_m(x)$ and, as this function is convex, that critical point is the only global minimizer.
Therefore, the production that minimizes the average cost is: $x = 6$.
Finally, substituting in the average cost function, the minimum average cost will be:
$C_m(6) = \dfrac{72}{6} + 9 + 2.6 = 12 + 9 + 12 = 33$

(4) **Let the function** $f(x) = \begin{cases} ax+3 & if\ x<1 \\ -x^2+2ax+b & if\ \ x \geqslant 1 \end{cases}$ **and let us consider the interval** $[-1,3]$ **. Then:**

(a) Find a and b in order that $f(x)$ satisfies the hypothesis (or initial conditions) of Lagrange's theorem (or mean value theorem) in that interval.

(b) For those values a, b find the intermediate value or values c in such a way that the thesis (or conclusion) of this theorem is satisfied.

Hint for both parts: state the mean value theorem.

0.6 points part a); 0.4 points part b)

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a) We need to set the continuity and derivability at $x=1$.
For that reason, as $\lim_{x \longrightarrow 1^-} f(x) = a+3$, $f(1) = \lim_{x \longrightarrow 1^+} f(x) = -1+2a+b$
it can be deduced that the function will be continuous at that point when:
$a+3 = -1+2a+b \Longleftrightarrow a+b=4$.
On the other hand, supposing the the function is continuous at $x=1$, it will have a derivative at that point if:
$a = f\,'_-(1) = f\,'_+(1) = -2+2a \Longleftrightarrow a=2$.
So the function will be continuous and derivable at $x=1$ when $a=b=2$.

b) By the mean value theorem we know that:
There exists $c \in (-1,3) : f(3) - f(-1) = f\,'(c)(3-(-1))$.
Taking into account that $a=b=2$, the former equation is equivalent to $(-9+12+2)-(-2+3) = 4f'(c)$.
In other words: $f'(c) = 1$.
When $x \leq 1, f\,'(x) = 2 \neq 1$, so it is not possible that $c \leq 1$.
when $x > 1, f\,'(x) = -2x+4 = 1 \Longleftrightarrow x = \frac{3}{2}$.
So the only possible value is $c = \frac{3}{2}$.

(5) **Given a continuous, concave function** $f : [0,3] \longrightarrow \mathbb{R}$ **and satisfying:** $f(0) = 2,\ f(1) = 3,\ f(3) = 0,\ f'(1) = 0$. **Then:**

(a) Draw approximately the set $A = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq f(x)\}$ and find the maximal and minimal elements, the maximum and the minimum of A if they exist.

(b) Calculate the best lower (or by default) approximation of the area of the given set.

Hint for a: Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

Hint for b: consider the segments that unite the points $(0, 2), (1, 3)$ and $(3, 0)$.

0.6 points part a); 0.4 points part b)

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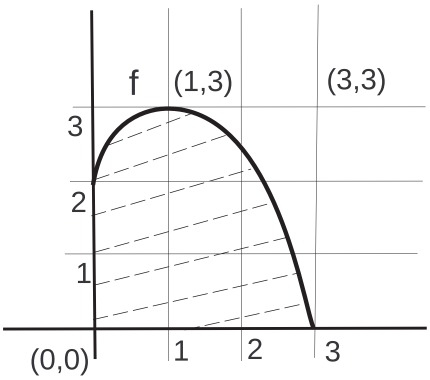
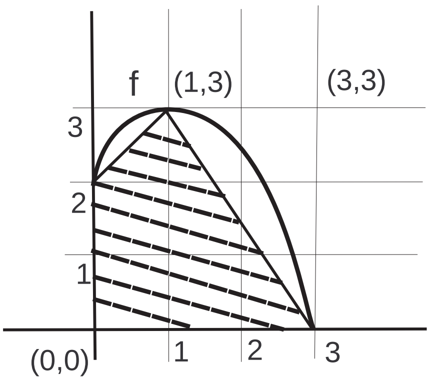
a) As $f(x)$ is concave and $f'(1) = 0$, it means that the function is increasing on the interval $[0, 1]$ and decreasing on the interval $[1, 3]$. So, the drawing of A will be, approximately, this way:



by this graph, the Pareto order describes the set in the following way:
there is not a maximum of A, maximal points$(A) = \{(x, f(x)) : 1 \leq x \leq 3\}$.
minimum(A) = minimal points$(A) = \{(0, 0)\}$.

b) As the function is concave, the segment $h_1(x)$ that unites the points $(0, 2)$ and $(1, 3)$ remains below the graph of the function, as the next figure shows:



So, as $h_1(x) = 2 + x$, it can be deduced that:

$\int_0^1 h_1(x)dx = \frac{5}{2} \leq \int_0^1 f(x)dx$

Analogously, the segment $h_2(x)$ that unites the points $(1, 3)$ and $(3, 0)$ also remains below the graph of the function. So, as $h_2(x) = (\frac{-3}{2})(x - 3)$, it can be deduced that:

$\int_1^2 h_2(x)dx = 3 \leq \int_1^2 f(x)dx$

So $\frac{5}{2} + 3 = \frac{11}{2} \leq$ Area (A)

(6) **Given the function** $h(x) = \dfrac{\ln x}{x^3}$, **if** $x > 0$, **then:**

(a) Find the primitive of $h(x)$ whose value at $x = 1$ is 1.

(b) Let g be a continuous function such that $h(x) \le g(x) \le 2h(x)$ if $x \geqslant 1$ and let $G(x) = \int_1^x g(t)dt$.

Find the best approximation of L, being $y = L$ the horizontal asymptote of $G(x)$.

Hint for b: first of all, prove the existence of L and compute its approximate value, supposing that $\lim_{x\longrightarrow\infty} \int_1^x h(t)dt = \frac{1}{4}$ and proving / using the monotonicity of $G(x)$.

Secondly, check that $\lim_{x\longrightarrow\infty} \int_1^x h(t)dt = \frac{1}{4}$

0.5 points part a); 0.5 points part b)

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a) Let $H(x) = \int \dfrac{\ln x}{x^3} dx$ be the indefinite primitive of $h(x)$.

Integrating by parts, and calling $f'(x) = x^{-3}, g(x) = \ln x$, you obtain:

$\int x^{-3} \ln x dx = \dfrac{x^{-2}}{-2} \ln x - \int \dfrac{x^{-2}}{-2} x^{-1} dx = \dfrac{x^{-2}}{-2} \ln x + \frac{1}{2} \int x^{-3} dx =$

$= \dfrac{x^{-2}}{-2} \ln x + \frac{1}{2}(\dfrac{x^{-2}}{-2}) + C = (\frac{-1}{4})x^{-2}(2\ln x + 1) + C$

And now, as $H_1(1) = \frac{-1}{4} + C = 1 \Longrightarrow C = \frac{5}{4}$.

So we have $H_1(x) = (\frac{-1}{4})x^{-2}(2\ln x + 1) + \frac{5}{4}$.

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b) As $\int_1^x h(t)dt \le \int_1^x g(t)dt \le 2\int_1^x h(t)$, and as $\lim_{x\longrightarrow\infty} \int_1^x h(t)dt = \frac{1}{4}$,

it follows that $L = \lim_{x\longrightarrow\infty} \int_1^x g(t)dt$, if it exists, it will satisfy $\frac{1}{4} \le L \le \frac{1}{2}$.

So $L = \frac{3}{8}$ will be the best approximation, with a maximum error of $\frac{1}{8}$.

On the other hand, as $G(x) = \int_1^x g(t)dt$ is increasing, because its derivative, $g(x)$, is positive, and also $G(x)$ is upperly bounded, it can be deduced that $L = \lim_{x\longrightarrow\infty} \int_1^x g(t)dt$ exist.

To prove the hint, as $H_2(x) = \int_1^x h(t)dt$ is the primitive of $h(x)$ that takes the value 0 in $x = 1$, then the mentioned primitive is exactly equal to the other primitive $H(x)$ obtained in part a), except in one unit.

So, $H_2(x) = (\frac{-1}{4})x^{-2}(2\ln x + 1) + \frac{1}{4}$. And now, $\lim_{x\longrightarrow\infty} \int_1^x h(t)dt =$

$= \lim_{x\longrightarrow\infty} H_2(x) = \frac{1}{4} + (\frac{-1}{4}) \lim_{x\longrightarrow\infty} (2\ln x + 1)/x^2$ =(applying the L'Hopital's rule)=

$= \frac{1}{4} + (\frac{-1}{4}) \lim_{x\longrightarrow\infty} \dfrac{2/x}{2x} = \frac{1}{4}$.

The following graphs of $g(x)$ and $G(x)$ can help to understand the situation: