Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **January 20^{th}, 2016**

Exam time: 2 hours.

LAST NAME: **FIRST NAME:**

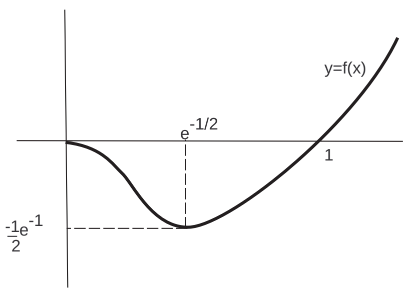
ID: **DEGREE:** **GROUP:**

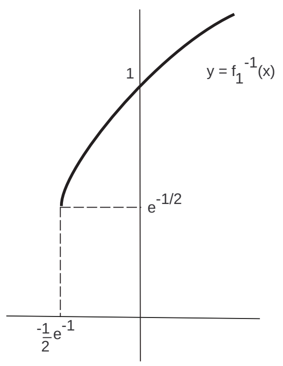
1. **Consider the function $f(x) = x^2 \ln x$. Then:**
   (a) draw the graph of the function, obtaining firstly its domain, the intervals where $f(x)$ increases and decreases, its global extrema (if they exist), asymptotes and range.
   (b) consider the new function $f_1(x) = f(x)$ (only defined on the interval where $f(x)$ is increasing). Find the domain, the range and the intervals of concavity/convexity of $f_1^{-1}(x)$. Sketch the graph of this function.
   Hint 1: Study the concavity/convexity of f_1 to find the concavity/convexity of $f_1^{-1}(x)$.
   Hint 2: Don't try to calculate the analytic expression of $f_1^{-1}(x)$.

   Part (a) 0.6 points; Part (b) 0.4 points.

(a) The domain of the function is $\{x : x > 0\} = (0, \infty)$.
On the other hand, since $f'(x) = 2x \ln x + x = x(2\ln x + 1)$, we know that f is decreasing in $(0, e^{-1/2}]$ and increasing in $[e^{-1/2}, \infty)$, so $1 + 2\ln(x) = 0 \iff \ln x = -\frac{1}{2} \iff x = e^{-1/2}$ and since the logarithm is an increasing function, then we have $1 + 2\ln(x) < 0$ if $x < e^{-1/2}$ (or, in other words, $f'(x) < 0$ in the first interval); and $1 + 2\ln(x) > 0$ if $x > e^{-1/2}$ (or, $f'(x) > 0$ in the second interval).

Regarding asymptotes, because the function is continuous in its domain we can only look for a vertical asymptote at 0^+ :
$\lim_{x \longrightarrow 0^+} f(x) = \lim_{x \longrightarrow 0^+} \dfrac{\ln(x)}{1/x^2} =$ (applying L'Hospital's Rule)$= \lim_{x \longrightarrow 0^+} \dfrac{1/x}{-2/x^3} = 0$; so the function doesn't have a vertical asymptote. Now, we study the behaviour of the function at ∞ :
$\lim_{x \longrightarrow \infty} f(x) = \infty$ and $\lim_{x \longrightarrow \infty} \dfrac{f(x)}{x} = \infty$; this means that there aren't any horizontal or oblique asymptotes.
Then, the function has a global minimum at $x = e^{-1/2}$, whose value is:
$f(e^{-1/2}) = e^{-2/2} \ln(e^{-1/2}) = -\frac{1}{2}e^{-1}$ and the range of the function is: $[-\frac{1}{2}e^{-1}, \infty)$.
To conclude, the graph of f is approximately as in the first figure.





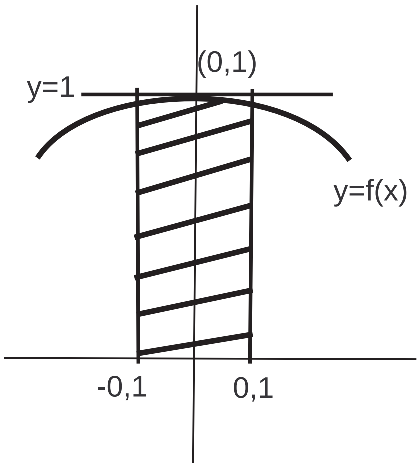
(b) We have defined $f_1 = f : [e^{-1/2}, \infty) \longrightarrow [-\frac{1}{2}e^{-1}, \infty)$ so, it is an increasing bijective function. Then, $f_1^{-1} : [-\frac{1}{2}e^{-1}, \infty) \longrightarrow [e^{-1/2}, \infty)$ is also increasing and bijective.
On the other hand, f_1 is convex, since $f''(x) = 2\ln x + 1 + x \cdot \frac{2}{x} = 2\ln x + 3$, and because $f''(x)$ is increasing, if $x > e^{-1/2} \Longrightarrow f''(x) > 2\ln e^{-1/2} + 3 = 2 > 0$, we can deduce that $f_1^{-1}(x)$ is concave, using the symmetry of the inverse function with respect to the first bisector line.
We conclude that the graph of f_1^{-1} is represented approximately as in the second figure.

2. **Let $y = f(x)$ be an implicit function defined by the equation $4x^2 + y^2 + y^6 = 2$, in a neighborhood of the point $x = 0, y = 1$. Then:**
   (a) find the tangent line of $f(x)$ at $x = 0$, and prove that $f(x)$ is concave near the point.
   (b) sketch the graph of the function around $x = 0$ and calculate approximately the area of the region bounded by the graph of $f(x)$, the x-axis, and the vertical lines $x = -\delta, x = \delta$, for a small $\delta > 0$. Is the approximation greater or smaller than the real area?
   Hint for (b): If you didn't find the asked tangent line, consider instead $y = 1 + mx$.

1 point

(a) Firstly, we have to calculate the first and second order implicit derivatives of the equation.
First derivative:
$8x + 2yy' + 6y^5y' = 8x + (2y + 6y^5)y' = 0$
then, we evaluate this at $x = 0$, $y(0) = 1$ to obtain $y'(0) = f'(0) = 0$.
Second derivative:
$8 + (2y' + 30y^4y')y' + (2y + 6y^5)y'' = 0$
and we evaluate at $y(0) = 1$, $y'(0) = 0$ to deduce that $y''(0) = f''(0) = -1$.
Hence, the equation of the tangent line is:
$y - 1 = 0(x - 0)$, or, $y = 1$.
Obiously, the implicit function is locally concave since, $f''(0) < 0$.

(b) Since the graph of f will be underneath the tangent line $y = 1$, a sketch of this in a neighbourhood of the point $x = 0$, is approximately as follows:



Furthermore, because the function is positive near the point $x = 0$, the area will be the integral $\int_{-\delta}^{\delta} f(x)dx$, where we approximately obtain the same result if we exchange $f(x)$ with the tangent line $y = 1$. This means, $\int_{-\delta}^{\delta} f(x) \approx \int_{-\delta}^{\delta} 1dx = 2\delta$.
Since the function is concave, its graph is underneath the tangent line and the approximate integral is greater.
Note: If you use the tangent line $y = 1 + mx$, the result is equal because, the area of a rectangle whose base is 2δ and height 1, is exactly the same as the area of a trapezium with the same base and average height 1.

3. **Let** $C'(x) = 0.04x + 2$ **and** $I'(x) = -0.16x + 102$ **be the marginal cost and revenue functions of a monopolistic firm, with** $x \geqslant 0$ **the number of produced units of a certain kind of goods. Then:**
   (a) find the production that maximizes the profit. For this level of production, what's the additional **approximate** profit of producing one less unit?
   (b) knowing that the cost of producing 100 units is 600 monetary units, find the production that minimizes the average cost. For this level of production, what's the additional **approximate** profit of producing one more unit?

   Part (a) 0.4 points; Part (b) 0.6 points.

(a) Firstly, we have to calculate the first and second derivative of B :
$B'(x) = I'(x) - C'(x) = -0.16x + 102 - (0.04x + 2) = -0.2x + 100; B''(x) = -0.2 < 0$
so we notice that B has only one critical point at $x = 500$ and since B is a concave function, then the critical point is a global maximizer.
With this level of production, the additional profit of producing one more or one less unit will be, approximately 0, since:
$B(501) - B(500) \approx B'(500) = 0$, $B(499) - B(500) \approx B'(500) = 0$.

(b) The cost function is $C(x) = 0,02x^2 + 2x + C_0$.
Since $C(100) = 0.02 \cdot 100^2 + 2 \cdot 100 + C_0 = 600 \implies C_0 = 200$, the average cost function is $C_m(x) = \dfrac{C(x)}{x} = \dfrac{200}{x} + 2 + 0.02x$.
If we calculate its first and second order derivative function:
$C'_m(x) = \dfrac{-200}{x^2} + 0.02; \ C''_m(x) = \dfrac{400}{x^3} > 0$
we can see that $x = \sqrt{\dfrac{200}{0.02}} = 100$ is the only critical point and because $C_m(x)$ is a convex function, that critical point is the only global minimizer.
Therefore, the level of production that minimizes the average cost is $x = 100$.
For this level of production, the additional profit of producing an extra unit will be approximately 80, since:
$B(101) - B(100) \approx B'(100) = -20 + 100 = 80$.

4. **Consider the function** $f(x) = xe^x$. **Then:**

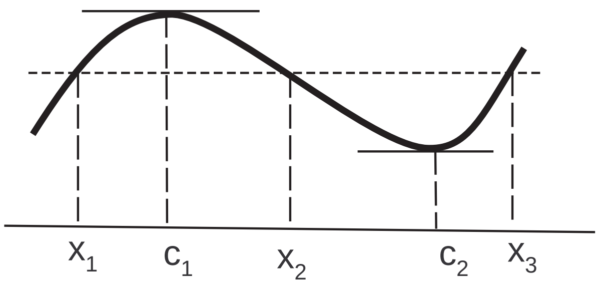
(a) state Rolle's theorem and use it to prove that there aren't three different numbers $x_1 < x_2 < x_3$ such that $f(x_1) = f(x_2) = f(x_3)$. What happens if we change $f(x)$ for a convex and differentiable function $g(x)$?

Hint : Apply Rolle's theorem in two different intervals. Also, we consider convexity as strict convexity.

(b) study if $f(x)$ is convex, find Taylor's polynomial of second order of $f(x)$ at $x = 0$ and calculate the approximate value of $f(\frac{1}{4})$.

Part (a) 0.6 points; Part (b) 0.4 points.

(a) Applying Rolle's theorem to f on the interval $[x_1, x_2]$, we can deduce the existence of $c_1 \in (x_1, x_2)$ such that $f\,'(c_1) = 0$.
Also, Applying Rolle's theorem to f on the interval $[x_2, x_3]$, we deduce the existence of $c_2 \in (x_2, x_3)$ such that $f\,'(c_2) = 0$.
Since, $f\,'(x) = e^x + xe^x = (x+1)e^x$, the only zero of the function $f'(x)$ is $x = -1$.
Therefore, f can only take the same value (image) twice.
Similarly, a convex function can only have a unique critical point, so it is impossible that there are three different points $x_1 < x_2 < x_3$ such that $g\,(x_1) = g(x_2) = g(x_3)$.
Approximately, we can sketch a function having the same value three times as follows:



(b) Since $f''(x) = e^x + (x+1)e^x = (x+2)e^x$, we can deduce that $f(x)$ is not convex on $(-\infty, -2)$.
Furthermore, since $f(0) = 0, f\,'(0) = 1, f''(0) = 2$, we deduce that $P(x) = x + x^2$.
Consequently, $f(\frac{1}{4}) \approx P(\frac{1}{4}) = 5/16$.

5. **Given the set** $A = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq 2 - |x|)\}$. **Then:**

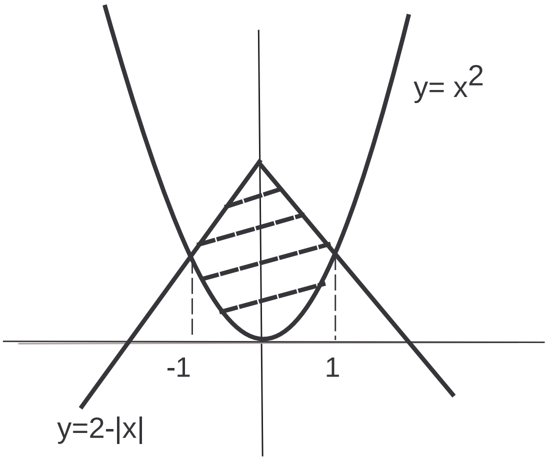
(a) draw the set A and find, if they exist, the maximal/minimal elements, the maximum and the minimum of A.

(b) calculate the area enclosed by the set A. What's the area enclosed by B= $\{(x,y) \in \mathbb{R}^2 : x^2 + 1 \leq y \leq 3 - |x|)\}$?

Hint : Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1$ and $y_0 \leq y_1$.

Part (a) 0.6 points; Part (b) 0.4 points.

(a) Since the set A is symmetrical with respect to the y-axis, we can only focus our attention on $x \geqslant 0$. For those values, $(x,y) \in A$ if $f(x) = x^2 \leq y \leq 2 - x = g(x)$.
The graphs of both functions $y = x^2, y = 2 - x$ intercept at $x = 1$.
Therefore, the sketch of A will be:



Since, $g(x)$ is decreasing on $[0,1]$ and $f(x)$ is also decreasing on $[-1,0]$,
Pareto's order describes the special points in the set as:
$maximum(A)$ doesn't exist, $\{\text{maximals}(A)\} = \{(x, 2-x) : 0 \leq x \leq 1\}$.
$minimum(A)$ doesn't exist, $\{\text{minimals}(A)\} = \{(x, x^2) : -1 \leq x \leq 0\}$.

(b) As we have mention before, the area is going to be twice the area of the set on the right to the y-axis. So,
$A = 2\int_0^1 (g(x) - f(x))dx = 2\int_0^1 (2 - x - x^2)dx = 2[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 = \frac{7}{3}$ square units.
On the other hand, B is just one unit vertical translation upwards of the set A, So the area of B is the same as that of A.

6. **Given the function** $f(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$, **if** $x > 0$, **then:**

   (a) find the primitive of $f(x)$ whose value at $x = 1$ is equal to 0.

   (b) suppose g is a continuous function such that $g(x) \geqslant 1 + \frac{1}{2}\sqrt{x}$ if $x \geqslant 1$. Calculate the asymptotes, if they exist, of the function $G(x) = \int\limits_1^x g(t)dt$.

   Hint for (b): First, prove that $\lim\limits_{x \longrightarrow \infty} G(x) = \infty$.

1 point

(a) Let $F(x) = \int \dfrac{e^{\sqrt{x}}}{\sqrt{x}} dx$ being the general primitive function of $f(x)$.
Making the change of variable $x = t^2, dx = 2tdt$, we obtain:
$\int \dfrac{e^t}{t} 2tdt = 2\int e^t dt = 2e^t + C = 2e^{\sqrt{x}} + C$
And, since $F(1) = 2e + C = 0 \Longrightarrow C = -2e$ then $F(x) = 2e^{\sqrt{x}} - 2e$.

(b) As $g(x) \geqslant 1 + \frac{1}{2}\sqrt{x}$ when $x \geqslant 1 \Longrightarrow G(x) = \int\limits_1^x g(t)dt \geqslant \int\limits_1^x (1 + \frac{1}{2}\sqrt{t})dt \geqslant x - 1 \longrightarrow \infty$ if $x \longrightarrow \infty$, then $G(x)$ cannot have a horizontal asymptote.
Since $G(x)$ is continuous in its domain it cannot have a vertical asymptote and because $\lim\limits_{x \longrightarrow \infty} \dfrac{G(x)}{x} = \frac{\infty}{\infty}$ =(applying L'Hospital's Rule)$= \lim\limits_{x \longrightarrow \infty} \dfrac{G'(x)}{1} = \lim\limits_{x \longrightarrow \infty} g(x) = \infty$ it hasn't got an oblique asymptote either.