Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Sum
Score							

Department of Economics **Final Exam of Mathematics I** **January 20th, 2015**

Time: 2 hours

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

1. Consider the function $f(x) = \dfrac{\ln(1+2x)}{1+2x}$. You are asked to (10 points)

(a) Draw the graph of the function, obtaining firstly its domain, the intervals where f increases and decreases, its asymptotes, and image.

(b) Consider the functions $f_1(x) := f(x)$ defined just in the interval where f is increasing, and $f_2(x) := f(x)$ defined just in the interval where f is decreasing. Find the domains and the images of the functions f_1^{-1} and f_2^{-1}, and draw their graphs.

Suggestion: Do **not** try to compute the analytical expressions of f_1^{-1} and f_2^{-1}.

(a) The domain of f is $D = \{x \in \mathbb{R} : 1 + 2x > 0\} = (-\frac{1}{2}, \infty)$. The first derivative is

$$f'(x) = \frac{\frac{2}{1+2x}(1+2x) - 2\ln(1+2x)}{(1+2x)^2} = \frac{2(1-\ln(1+2x))}{(1+2x)^2}$$

One has $f'(x) = 0$ holds iff $\ln(1+2x) = 1$, that is, $1 + 2x = e$. So, the unique critical point is $x^* = \dfrac{e-1}{2}$.

$$f(x^*) = \frac{\ln(1+2x^*)}{1+2x^*} = \frac{\ln(1+e-1)}{1+e-1} = \frac{1}{e}$$

- $f'(x) \geq 0 \iff 1 \geq \ln(1+2x) \iff e \geq 1+2x \iff x^* \geq x$. So, f is increasing in $(-\frac{1}{2}, \frac{e-1}{2}]$
- $f'(x) \leq 0 \iff 1 \leq \ln(1+2x) \iff e \leq 1+2x \iff x^* \leq x$. So, f is decreasing in $[\frac{e-1}{2}, +\infty)$

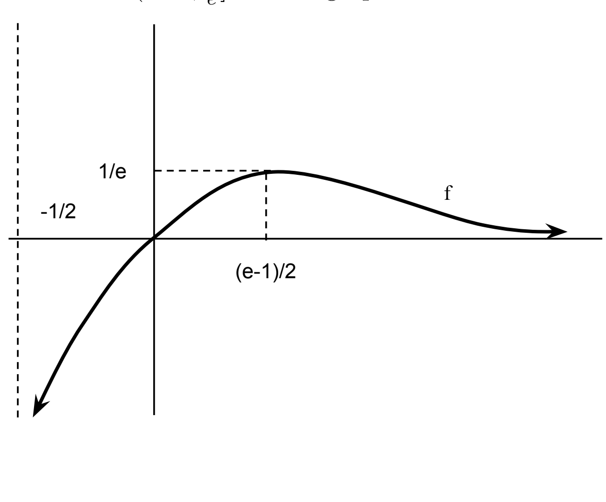
Consequently, f has a local maximum in the point $(x^*, f(x^*)) = (\frac{e-1}{2}, \frac{1}{e})$. In fact, it is global.

Since f is continuous on its domain, we just study the asymptotes at $-\frac{1}{2}^+$ and $+\infty$.

$$\lim_{x\to -\frac{1}{2}^+} f(x) = \frac{-\infty}{0^+} = -\infty; \qquad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln(1+2x)}{1+2x} = \frac{\infty}{\infty} \overset{\text{L'Hôpital}}{=} \lim_{x\to+\infty} \frac{\frac{2}{1+2x}}{2} = 0;$$

Hence, f has a vertical asymptote in $x = -\frac{1}{2}$ (on the right) and an horizontal asymptote in $y = 0$ (in $+\infty$). There are no oblique asymptotes.

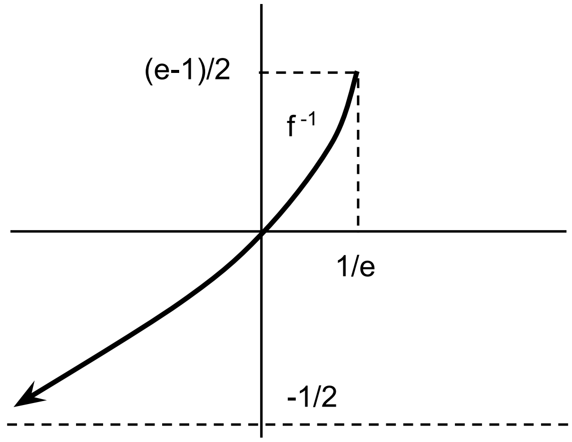
Then, the image of the function is $(-\infty, \frac{1}{e}]$ and its graph is as follows



(b) By definition $f_1(x) = f(x)$ for all $x \in (-\frac{1}{2}, \frac{e-1}{2}]$ and it is a bijective and increasing function. Hence,

$$f_1 : \left(-\frac{1}{2}, \frac{e-1}{2}\right] \longrightarrow \left(-\infty, \frac{1}{e}\right] \qquad \text{and so} \qquad f_1^{-1} : \left(-\infty, \frac{1}{e}\right] \longrightarrow \left(-\frac{1}{2}, \frac{e-1}{2}\right]$$

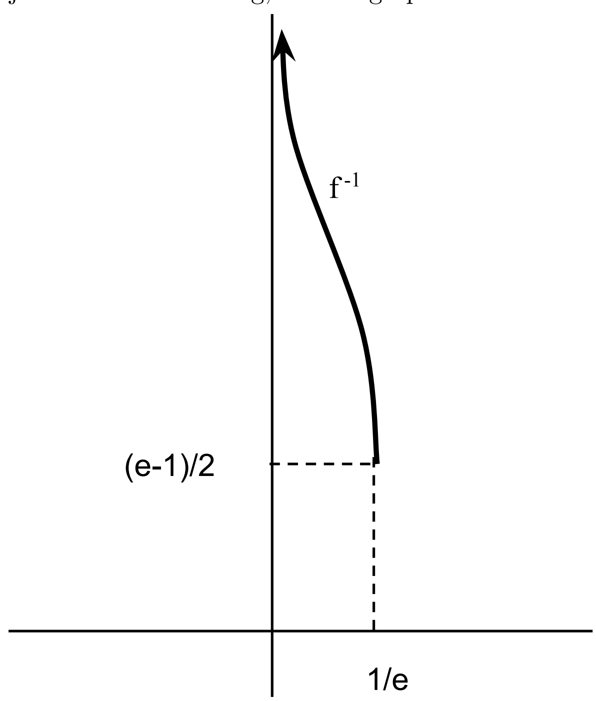
The function f_1^{-1} is also bijective and increasing, and its graph is



By definition $f_2(x) = f(x)$ for all $x \in [\frac{e-1}{2}, +\infty)$ and it is a bijective and decreasing function. Hence,

$$f_2 : \left[\frac{e-1}{2}, +\infty\right) \longrightarrow \left(0, \frac{1}{e}\right] \qquad \text{and so} \qquad f_2^{-1} : \left(0, \frac{1}{e}\right] \longrightarrow \left[\frac{e-1}{2}, +\infty\right)$$

The function f_2^{-1} is also bijective and decreasing, and its graph is



2. Given the function $y = f(x)$ implicitly defined by the equation $y + e^{x+y} = 0$ in a neighborhood of the point $x = 1$, $y = -1$, you are asked to (10 points)

(a) Find the second-order Taylor polynomial of $f(x)$ around $a = 1$. Use it to get an approximation of $f(0,9)$.

(b) Find the equation of the tangent line to f at the point $x = 1$. Draw a sketch of the graph of f around the point $x = 1$.
Hint: Use the fact that $f'(1) < 0$ and $f''(1) < 0$.

(a) Firstly, we compute the first and second derivatives of the function

$$\begin{aligned} f(x) + e^{x+f(x)} &= 0 \\ f'(x) + (1 + f'(x))e^{x+f(x)} &= 0 \\ f''(x) + f''(x)e^{x+f(x)} + (1 + f'(x))^2 e^{x+f(x)} &= 0 \end{aligned}$$

Next we substitute $x = 1$, $f(1) = -1$,

$$\begin{aligned} f'(1) + (1 + f'(1))e^0 &= 0 \\ f''(1) + f''(1)e^0 + (1 + f'(1))^2 e^0 &= 0 \end{aligned}$$

Consequently, $f'(1) = -\frac{1}{2}$ and $f''(1) = -\frac{1}{8}$.

So, the second-order Taylor polynomial around $a = 1$ is

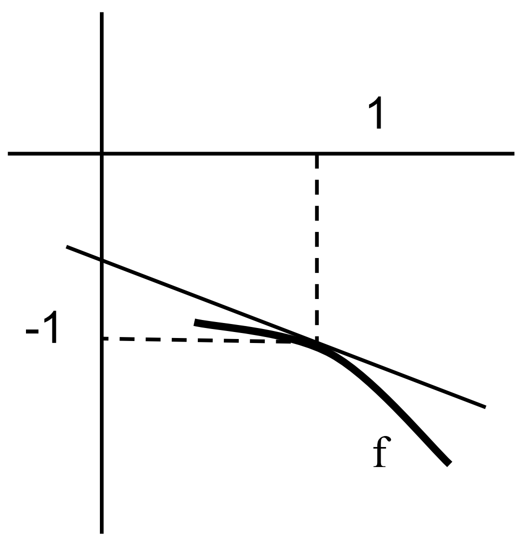
$$P(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = -1 - \frac{1}{2}(x-1) - \frac{1}{16}(x-1)^2$$

$$f(0,9) \approx P(0,9) = -1 + \frac{1}{2}(0,1) - \frac{1}{16}(0,1)^2 = \frac{-1600 + 80 - 1}{1600} = \frac{-1521}{1600}.$$

(b) Finally, the equation of the tangent line to f at the point $x = 1$ is

$$y = -1 - \frac{1}{2}(x-1) = \frac{-x-1}{2}$$

Since $f''(1) < 0$, the function f is concave in a neighborhood of $x = 1$. Hence, the graph of f lies below the given tangent line around the point $x = 1$.



3. Let $C(x) = C_0 + 40x + 0,04x^2$ be the cost function of a monopolistic firm, where $C_0 > 0$ represents the fixed costs and $x \geq 0$ is the output. The inverse demand function is given by $p(x) = 60 - 0,06x$. You are asked to (10 points)

(a) Find the price p^* that maximizes the benefit of the firm. Justify why it gives the maximum benefit.

(b) Find the value of C_0 such that the level of output that maximizes the benefit coincides with the level of output that minimizes the average costs. In such a case, which is the benefit? And the average cost?

(a) The benefit function is

$$B(x) = P(x)x - C(x) = 60x - 0,06x^2 - \left(C_0 + 40x + 0,04x^2\right) = -0,1x^2 + 20x - C_0.$$

One has

$$B'(x) = -0,2x + 20 \qquad \text{and} \qquad B''(x) = -0,2 < 0$$

B is a concave function and it has a unique critical point in $x^* = 100$, so that point is a global maximizer. The price associated to this level of output is $p^* = p(x^*) = p(100) = 60 - 0,06 \cdot 100 = 60 - 6 = 54$.

(b) The average cost function is

$$CM(x) = \frac{C(x)}{x} = \frac{C_0}{x} + 40 + 0,04x$$

One has

$$CM'(x) = \frac{-C_0}{x^2} + 0,04 \qquad \text{and} \qquad CM''(x) = \frac{2C_0}{x^3} > 0$$

CM is a convex function and it has a unique critical point in $\tilde{x} = \sqrt{\frac{C_0}{0,04}}$, so that point is a global minimizer.

By hypothesis, the level of output that maximizes the benefit coincides with the level of output that minimizes the average costs, that is, $x^* = \tilde{x}$ and so

$$\sqrt{\frac{C_0}{0,04}} = 100 \quad \Rightarrow \quad C_0 = 0,04 \cdot 100^2 = 400$$

For that value of C_0, the maximum benefit of the firm is

$$B(x^*) = B(100) = -0,1 \cdot 100^2 + 20 \cdot 100 - 400 = 600,$$

whereas the minimum average cost is

$$CM(\tilde{x}) = 4 + 40 + 4 = 48$$

4. Let $f : [0,3] \to \mathbb{R}$ be a continuous function in $[0,3]$ and twice derivable in $(0,3)$ such that

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 8.$$

You are asked to (10 points)

(a) Prove that there exist $c_1 \in (0,1)$ such that $f'(c_1) = 1$ and $c_2 \in (2,3)$ such that $f'(c_2) = 4$.

(b) Prove that there exists $c_3 \in (0,3)$ such that $1 < f''(c_3) < 3$.
Hint: Apply the Lagrange's Theorem to the appropriate function in the appropriate interval, and find a lower bound and an upper bound for $c_2 - c_1$.

(a) By applying the Lagrange Theorem to f in $[0,1]$, we get the existence of $c_1 \in (0,1)$ such that

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = 1.$$

Analogously, by applying the Lagrange Theorem to f in $[2,3]$, we get the existence of $c_2 \in (2,3)$ such that

$$f'(c_2) = \frac{f(3) - f(2)}{3 - 2} = 4.$$

(b) By applying the Lagrange Theorem to f' in $[c_1, c_2] \subset [0,3]$, we get the existence of $c_3 \in (c_1, c_2) \subset (0,3)$ such that

$$f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{3}{c_2 - c_1}.$$

Since $c_1 \in (0,1)$ and $c_2 \in (2,3)$, one has $-c_1 \in (-1,0)$ and so $1 < c_2 - c_1 < 3$. Hence,

$$1 > \frac{1}{c_2 - c_1} > \frac{1}{3} \text{ and so } 3 > \frac{3}{c_2 - c_1} > 1.$$

Consequently, $1 < f''(c_3) < 3$.

5. Consider the set $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}$ where f is an increasing function and convex in the interval $[2, 4]$ and it holds $f(2) = 5$, $f'(2) = 3$, $f(4) = 12$. You are asked to (10 points)

(a) Draw the set A and find, if they exist, the maximals, minimals, maximum and minimum points of A. Recall that the Pareto order is defined by $(x_0, y_0) \leq_P (x_1, y_1) \Leftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

(b) Find the best approximations (one from below and the other from above) of the area of the set A. Hint: Draw the graph of the function, the tangent line to f in $(2, f(2))$, and the straight line that crosses points $(2, f(2))$ and $(4, f(4))$.
Remark: the difference between both approximations is 1 unit area.

(a) Graph is shown in part (b). Since the function f is positive and increasing in $[2, 4]$, one has

$$\text{maximum}(A) = \text{maximals}(A) = \{(4, f(4))\}$$

$$\text{minimum}(A) = \text{minimals}(A) = \{(2, 0))\}$$

(b) Due to the convexity, the graph of the function lies above the tangent line to f in $(2, f(2))$, which is

$$y - 5 = 3(x - 2) \quad \Rightarrow \quad y = 3x - 1$$

On the other hand, also due to the convexity, the graph of the function lies below the straight line that crosses points $(2, f(2))$ and $(4, f(4))$, which is $y = 3.5x - 2$.

Hence, since f is positive and increasing, if

$$F := \text{area}(A) = \int_2^4 f(x)dx,$$

one has

$$F \;\geq\; F_b := \int_2^4 (3x - 1)dx = 3\frac{x^2}{2} - x|_2^4 = (24 - 4) - (6 - 2) = 16$$

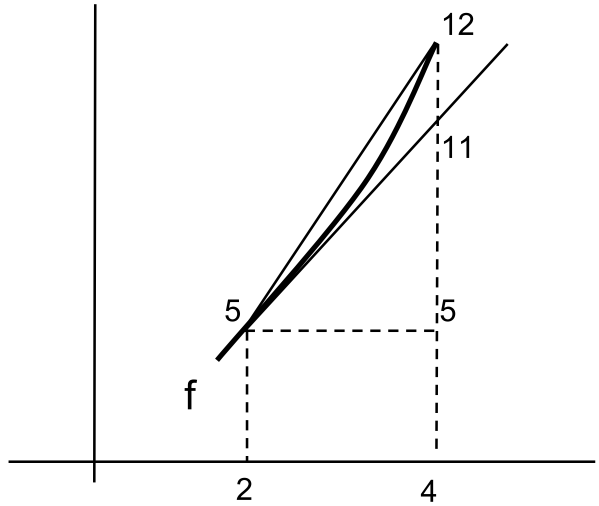
$$F \;\leq\; F_a := \int_2^4 (3.5x - 2)dx = 3.5\frac{x^2}{2} - 2x|_2^4 = (28 - 8) - (7 - 4) = 17$$

Another way to get these estimations is the following one. Observe that F_b is the area of the rectangle with width 2 (the length of the interval $[2, 4]$) and height 5 ($f(2)$) plus the area of the right triangle which is above the previous rectangle and has the same width, 2, and height 6 (the distance from $(4, 5)$ to $(4, 11)$). Hence,

$$F_b = 2 \cdot 5 + \frac{2 \cdot 6}{2} = 10 + 6 = 16$$

On the other hand, F_a is the area of the rectangle with width 2 (the length of the interval $[2, 4]$) and height 5 ($f(2)$) plus the area of the right triangle which is above the previous rectangle and has the same width, 2, and height 7 (the distance from $(4, 5)$ to $(4, 12)$). Hence,

$$F_a = 2 \cdot 5 + \frac{2 \cdot 7}{2} = 10 + 7 = 17$$



12
11
5
5
f
2
4

6. Consider the function

$$F(x) = \int_3^x \frac{2t-7}{t^2-t-2}dt$$

defined in $[3,+\infty)$. You are asked to (10 points)

(a) Find the local extreme points of F and classify them.

(b) Find the value of $F(4)$ and justify whether it is positive or negative.

Remark: Statements (a) and (b) are independent each other.

(a) By applying the Fundamental Theorem of Integral Calculus, one has $F'(x) = \dfrac{2x-7}{x^2-x-2}$.

Hence, $F'(x) = 0$ if and only if $2x - 7 = 0$ and so, $x^* = \dfrac{7}{2}$ is the unique critical point.

Observe that the points -1 and 2 where F' is not defined and so F would not be differentiable at those points, are not critical points since we are assuming that F is just defined at $[3,+\infty)$.

Now, we study the second derivative of F at x^* to classify that point.

$$F''(x) = \frac{2(x^2-x-2)-(2x-7)(2x-1)}{(x^2-x-2)^2} = \frac{-2x^2+14x-11}{(x^2-x-2)^2}$$

$$F''(x^*) = \frac{-2(49/4)+14(7/2)-11}{((49/4)-(7/2)-2)^2} = \frac{(-49+98-22)/2}{((49-14-22)/4)^2} = \frac{27/2}{(27/4)^2} = \frac{8}{27} > 0$$

Hence, F attains a local minimum in $x^* = \dfrac{7}{2}$.

(b) Since $t^2 - t - 2 = (t+1)(t-2)$, then

$$\frac{2t-7}{t^2-t-2} = \frac{A}{t+1} + \frac{B}{t-2} \qquad \Rightarrow \qquad 2t-7 = A(t-2)+B(t+1).$$

If we susbtitute $t = 2$ then we get $-3 = 3B$ and so $B = -1$. Analogously, if we substitute $t = -1$ we get $-9 = -3A$ and so $A = 3$. Hence,

$$F(4) = \int_3^4 \frac{2t-7}{t^2-t-2}dt = \int_3^4 \frac{3}{t-2}dt + \int_3^4 \frac{-1}{t+1}dt = [3\ln(t+1) - \ln(t-2)]_3^4 =$$

$$= (3\ln(5) - \ln(2)) - (3\ln(4) - \ln(1)) = 3(\ln(5) - \ln(4)) - \ln(2) = 3\ln\left(\frac{5}{4}\right) - \ln(2) =$$

$$= \ln\left(\frac{5}{4}\right)^3 - \ln(2) = \ln\left(\frac{125}{64}\right) - \ln(2) = \ln\left(\frac{125}{128}\right) < 0$$

since $\frac{125}{128} < 1$.