UNIVERSIDAD CARLOS III DE MADRID

Exercise	1	2	3	4	5	6	Sum
Score							

Department of Economics   Final Exam of Mathematics I   January 21st, 2014
Time: 2 hours

LAST NAME:		FIRST NAME:
ID:	DEGREE:	GROUP:

$\boxed{1}$ Let f be the function defined by $f(x) = \sqrt{25 + e^{2x}}$. You are asked to (1 point)

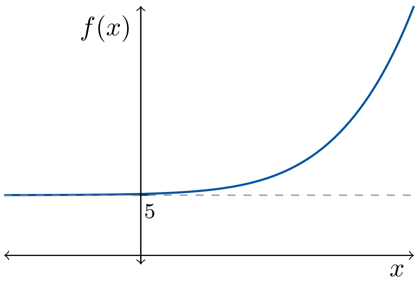
a) Draw the graph of the function, obtaining its domain, image, the intervals where f increases and decreases, and its asymptotes.

b) Find the analytical expression of $f^{-1}(x)$, specifying its domain, image, and asymptotes. Draw the graph of $f^{-1}(x)$.

Solution.

a) The domain of f is the interval $(-\infty, \infty)$.

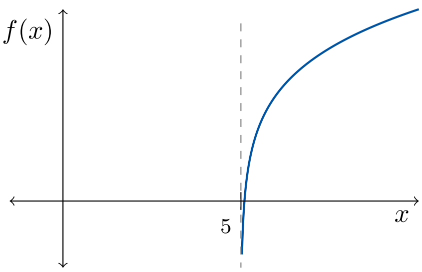
On the other hand, because $f'(x) = \dfrac{2e^{2x}}{2\sqrt{25 + e^{2x}}} > 0$, we know that f increasing and continuous in its domain. Even more, since $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to+\infty} f(x) = \infty$, we also know that the image of f is $(5, \infty)$, and that there do not exist vertical asymptotes. In order to compute the horizontal and slant asymptotes, it is enough to notice that $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to+\infty} \frac{f(x)}{x} = \infty$. Hence, the function has a horizontal asymptote in $-\infty$, while it has not asymptote in ∞. To conclude, the graph of f is:



b) Consider the equation $y = \sqrt{25 + e^{2x}}$, where $x \in \mathbb{R}$ and $y > 5$. Then,

$$y = \sqrt{25 + e^{2x}} \Leftrightarrow y^2 = 25 + e^{2x} \Leftrightarrow y^2 - 25 = e^{2x} \Leftrightarrow \ln(y^2 - 25) = 2x \Leftrightarrow x = \frac{1}{2}\ln(y^2 - 25).$$

Thus, $f^{-1}(x) = \frac{1}{2}\ln(x^2 - 25)$. The domain of $f^{-1}(x)$ is $(5, \infty)$, and its image is $\mathbb{R}$. Because of the graph of $f^{-1}(x)$ is symmetric to the graph of $f(x)$, $f^{-1}(x)$ is increasing, with vertical asymptote at $x = 5$ on the right of 5, and has not horizontal asymptote in ∞. Therefore, the graph of f^{-1} is



$\boxed{2}$ Given the function $y = f(x)$, implicitly defined by the equation $x + y^3 = x^3 + y$ in a neighborhood of $x = -1, y = 1$, you are asked to (1 point)

a) Write the second-order Taylor polynomial of $f(x)$ around $a = -1$. Use it to approximate the value of $f(-0{,}9)$.

b) Find the expression of tangent line of f at the point $x = -1$. Draw a sketch of the graph of f around the point $x = -1$. (*Hint: In order to represent f, it is enough to obtain the tangent line and use the fact that $f''(-1) < 0$*).

Solution.

a) Firstly, we compute the first and second derivatives of the function

$$1 + 3y^2y' = 3x^2 + y'$$

and

$$6y(y')^2 + 3y^2y'' = 6x + y''.$$

Then, we substitute the point $x = -1, y = 1$ to obtain that $f(-1) = 1$ (given in the statement of the problem), $f'(-1) = 1$, and $f''(-1) = -6$. Hence, the second-order Taylor polynomial around $a = -1$ is

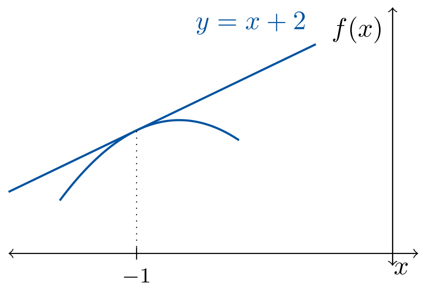
$$P(x) = 1 + (x + 1) + \frac{(-6)}{2}(x + 1)^2.$$

Therefore, we have that $f(-0{,}9) \approx P(-0{,}9) = 1 + 0{,}1 - 3(0{,}1)^2 = 1{,}07$.

b) The equation of the tangent line is given by

$$y = x + 2.$$

Besides, since $f''(-1) = -6 < 0$, the function f is concave in a neighborhood of $x = -1$. Therefore, the graph of f is, around $x = -1$, underneath the tangent line. The following picture illustrates the situation.



$\boxed{3}$ A farmer sells 400 tons of rice at a price of 200 euros each ton. This farmer estimates that his sales would increase by 100 tons if the price declines by 10 euros per ton. (1 point)

a) Find the inverse demand and marginal revenue functions. Compare them. (*Hint: We assume that the inverse demand function, $p = f(x)$, is linear, that is, $f(x) = ax + b$*).

b) If the fixed cost of producing x tons is C_0 euros, and the marginal cost is constant and equal to 40, obtain the fixed cost such that, when the farmer produces the amount that maximizes his profit, the average cost is 50 euros.

Solution.

a) Let $p(x) = ax + b$ be the inverse demand function. Notice that

(i) The farmer sells 400 tons at 200 euros each, i.e., $200 = 400a + b$,.

(ii) The farmer decreases 10 euros the price of the rice and increases the sales by 100, i.e., $190 = 500a + b$.

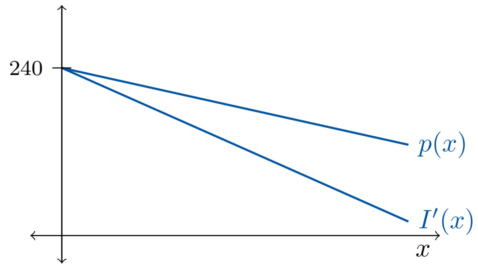
From these two facts we find that $a = -\frac{1}{10}$ and $b = 240$. Therefore, the inverse demand function is

$$p = -\frac{1}{10}x + 240.$$

The revenue function is given by $I(x) = p(x)x = -\frac{1}{10}x^2 + 240x$, which implies that the marginal revenue function is

$$I'(x) = -\frac{1}{5}x + 240.$$

As we can check, both the inverse demand and marginal revenue functions are straight lines. Both functions start from the point $(x, p) = (0, 240)$ and are decreasing, even though, the marginal revenue function is steeper than the inverse demand function. The following picture illustrates the situation.



b) First, the cost function is $C(x) = 40x + C_0$. Therefore, the profit function is $B(x) = I(x) - C(x) = -\frac{1}{10}x^2 + 200x - C_0$. Since the profit function is concave (because $B''(x) = -\frac{1}{5} < 0$), the critical point, if it exists, will be a global maximum. Hence,

$$B'(x) = -\frac{1}{5}x + 200 = 0 \equiv x = 1000.$$

On the other hand, since the fixed costs of producing 1000 must be 50 euros, we have that:

$$50 = \frac{C(1000)}{1000} = 40 + \frac{C_0}{1000} \quad \equiv \quad C_0 = 10000.$$

$\boxed{4}$ Let $f : [0,3] \longrightarrow \mathbb{R}$ be a function that is continuous in $[0,3]$ and differentiable in $(0,3)$, which satisfies that $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. You are asked to (1 point)

a) State the Mean Value Theorem (or Lagrange's Theorem).

b) Prove that there exist $c_1 \in (0,1)$ such that $f'(c_1) = 1$ and $c_2 \in (2,3)$ such that $f'(c_2) = 4$.

Solution.

a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function that is continuous in $[a,b]$ and differentiable in (a,b). Then, there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.

b)
- Applying the Mean Value Theorem to f in the interval $[0,1]$, we conclude the existence $c_1 \in (0,1)$ such that $f'(c_1) = \dfrac{f(1) - f(0)}{1-0} = 1$.
- Applying the Mean Value Theorem to f in the interval $[2,3]$, we conclude the existence $c_2 \in (2,3)$ such that $f'(c_2) = \dfrac{f(3) - f(2)}{3-2} = 4$.

$\boxed{5}$ Consider the set $A = \{(x, y) \in \mathbb{R}^2 : f(x) \leq y \leq g(x)\}$, where $f(x) = |x^2 - 1|$ and $g(x) = -2x^2 + 2$. You are asked to (1 point)

a) Draw the set A and find, if they exist, the maximal, minimal, maximum and minimum points of A.

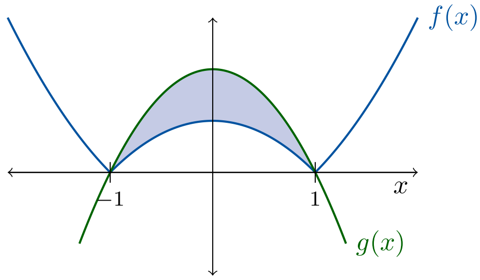
b) Compute the area of the region determined by the set A.

Hint: The Pareto ordering is given by $(x_0, y_0) \leq_P (x_1, y_1) \Leftrightarrow x_0 \leq x_1, y_0 \leq y_1$. **Solution.**

a) Notice that $g(x) = -2(x^2 - 1) = 2(1 - x^2)$ and $f(x)$ can be rewritten as follows

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq -1 \\ 1 - x^2 & \text{if } -1 \leq x \leq 1 \\ x^2 - 1 & \text{if } 1 \leq x \end{cases}$$

It happens that $f(x) = g(x)$ when $x \in \{-1,1\}$, in which case $f(-1) = g(-1) = f(1) = g(1) = 0$. Obviously, if $x < -1$ or $x > 1$ then $g(x) < 0 < f(x)$, and consequently (x, y) does not belong to the set A for any y. Lastly, if $-1 \leq x \leq 1$, both functions are concave parabola, they are also symmetric with respect to the vertical axis, and they cross each other at points $(0, 2)$ and $(0, 1)$. Thus, A is the following set



The maximal points of A are $\{(x, g(x)) : 0 \leq x \leq 1\}$. The set A has no maximum. Both the minimal and minimum points coincide at $(-1, 0)$.

b) The requested area, using the symmetry with respect to the vertical axis, is

$$\int_{-1}^{1} (g(x) - f(x))dx = 2\int_{0}^{1} \left[(-2x^2 + 2) - (1 - x^2)\right] dx = 2\int_{0}^{1} (-x^2 + 1)dx = 2\left[-\frac{1}{3}x^3 + x\right]_0^1$$
$$= 2\left(-\frac{1}{3} + 1\right) = \frac{4}{3}\, u^2.$$

$\boxed{6}$ Given the function $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$. You are asked to (1 point)

a) Compute, in case it exists, the primitive $F(x)$ of f such that $F(1) = -1$. (*Hint: Make the following change of variable* $x = t^2$).

b) Obtain the equation of the tangent line to function $F(x)$ at $x = 4$.

Solution.

a) We make the change $x = t^2$, so $\sqrt{x} = t$ and $dx = 2tdt$. Then, the primitive of f is:

$$\int f(x)dx = \int \frac{t}{1+t} 2tdt = 2\int \frac{t^2}{1+t}dt = 2\int \left(\frac{t^2-1}{t+1} + \frac{1}{1+t}\right) dt = 2\int \left(t - 1 + \frac{1}{1+t}\right) dt$$
$$= 2\left(\frac{t^2}{2} - t + \ln(1+t) + C\right) = 2\left(\frac{x}{2} - \sqrt{x} + \ln(1+\sqrt{x}) + C\right).$$

Finally, since $F(1) = -1$, we have that $2\left(\dfrac{1}{2} - 1 + \ln 2 + C\right) = -1$ and $C = -\ln 2$. Therefore, $F(x) = 2\left(\frac{x}{2} - \sqrt{x} + \ln(1+\sqrt{x}) - \ln 2\right)$.

b) In application of the Fundamental Theorem of Calculus, it holds that $F'(x) = f(x)$. And then $F'(4) = f(4) = \frac{2}{3}$. On the other hand, $F(4) = 2\left(\dfrac{4}{2} - \sqrt{4} + \ln(1+\sqrt{4}) - \ln 2\right) = 2(\ln 3 - \ln 2) = \ln \dfrac{9}{4}$. Hence, the equation of the tangent line is $y - F(4) = F'(4)(x-4)$, that is, $y - \ln\frac{9}{4} = \frac{2}{3}(x-4)$.