Question	1	2	3	4	5	6	Total
Grade							

Economics Department **Final Exam Mathematics I** **January 16, 2013**

Total Length: 2 hours.

SURNAME:		**NAME:**
DNI:	**Degree:**	**Group:**

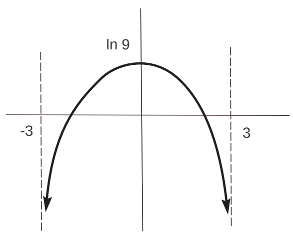
(1) **Let f be the function defined as $f(x) = \ln(9 - x^2)$. We ask you to:**

(a) Draw the graph of the function, first finding the domain as well as, the intervals on which $f(x)$ increases and decreases, then the asymptotes and the range of $f(x)$.

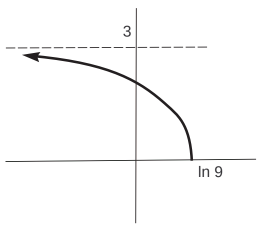
(b) Consider the function $f(x)$, restricted to the interval $[0, 3)$. Find the analytical expression of $f^{-1}(x)$, its domain and range, and sketch the graph of $f^{-1}(x)$.

1 point

(a) The domain of the previous function is the interval $(-3, 3)$.
On the other hand, as $f\,'(x) = \dfrac{-2x}{9 - x^2}$,you can deduce that f is increasing in the interval $(-3, 0]$ and decreasing in the interval $[0, 3)$.
In order to compute the asymptotes of f, it is enough to take into account that: $\lim\limits_{x \longrightarrow -3^+} f(x) = \lim\limits_{x \longrightarrow 3^-} f(x) = \ln(0^+) = -\infty$, so the function has vertical asymptotes at the points $x = -3, x = 3$.
On the other hand, as its domain is a bounded interval, it has no horizontal nor oblique asymptotes.
The range of the function is $(-\infty, f(0)] = (-\infty, \ln(9)]$, by the monotonicity of f and its vertical asymptotes. So, the graph of f will be, approximately, like this:



(b) Let us consider the equation $y = \ln(9 - x^2)$, where $0 \leq x < 3, -\infty < y \leq \ln 9$. Then,
$y = \ln(9 - x^2) \Longleftrightarrow e^y = 9 - x^2 \Longleftrightarrow x^2 = 9 - e^y \Longleftrightarrow x = \sqrt{9 - e^y}$.
So $f^{-1}(x) = \sqrt{9 - e^x}$.
The domain of $f^{-1}(x)$is $\underset{x \longrightarrow -\infty}{(-\infty, \ln 9]}$ and its range is $[0, 3)$.
As the graph of $f^{-1}(x)$ is the symmetric to the graph of $f(x)$, considering this function defined only on the interval $[0, 3)$, $f^{-1}(x)$ is a decreasing function, with asymptote $y = 3$ on $-\infty$, and reaching its absolute minimum at $x = \ln 9$, where the function will take the value 0. For these reasons, the graph of f^{-1} look approximately like this:



(2) **Given the function** $f(x) = e^x \ln(1-x)$**,we ask you to:**

(a) Find the secord order Taylor polynomial of $f(x)$, centered at $a = 0$, and use it to obtain an approximation of the value of $f(0,1)$.

(b) Find the equation of the tangent line to f at the point $x = 0$ and sketch the graph of f near the point $x = 0$.

Hint for b): in order to represent f, it is only necessary to find the tangent line and use the fact that $f"(0) < 0$.

1 point

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a) First of all, we compute the first and second derivative of the function:

$f\,'(x) = e^x[\ln(1-x) - \dfrac{1}{1-x}]$

$f"(x) = e^x[\ln(1-x) - \dfrac{2}{1-x} - \dfrac{1}{(1-x)^2}]$

Afterwards, we substitute on the point $x = 0$, and obtain that:

$f(0) = 0, f\,'(0) = -1, f"(0) = -3$

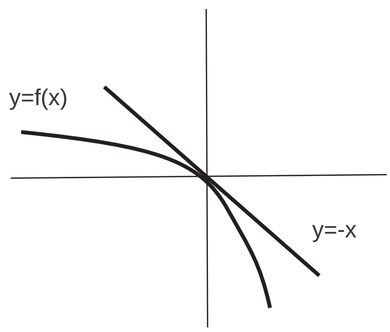
So the second order Taylor polynomial, centered at a=0, will be:

$P(x) = -x - \frac{3}{2}x^2$.

So, we have that $f(0,1) \approx P(0,1) = -0,1 - \frac{3}{2}(0,1)^2 = -0,1 - 0,015 = -0,115$.

b) The equation of the tangent line will be: $y = -x$. Moreover, as $f"(0) = -3 < 0$, the function f is concave near the point $x = 0$.

For these reasons the graph of f will be below the tangent line and it will look, near the point $x = 0$, approximately like that:



(3) **Let $C(x) = 25.000 + 450x + 0,03x^2$ and $p(x) = 550 - 0,02x$ be the cost and (inverse) demand functions, respectively, of a monopolistic firm. We ask you to:**

(a) Find the level of production x_0 and the price p_0 where the firm obtains its maximum profit. Find also the maximum profit.

(b) Find the level of production x_1 where the firm obtains its maximum mean profit (or by unit), i.e., the production that maximizes the function $B_{me}(x) = \dfrac{B(x)}{x}$.

Compare the behavior of the functions $B(x)$ and $\dfrac{B(x)}{x}$ in the interval $[x_0, x_1]$, looking at whether they are increasing or decreasing.

Remark for b): in order to simplify the operations, you may take 1,4 as an approximation of $\sqrt{2}$.

1 point

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a) The revenue function is $I(x) = 550x - 0,02x^2$, so the profit function will be:
$B(x) = I(x) - C(x) = -0,02x^2 + 550x - (0,03x^2 + 450x + 25.000) =$
$= -0,05x^2 + 100x - 25000.$
We observe that the profit function is concave ($B"(x) = -0,1 < 0$).
So, the critical point, if it exists, will be the unique absolute maximizer.
As $B'(x) = -0,1x + 100 = 0 \Longleftrightarrow x = 1.000.$
So that level of production is the one which maximizes the profit.
Analogously, the price which maximizes the profit function is $p = 550 - 0,02.1000 = 530.$
Finally, the maximum profit is $B(1.000) = -50.10^3 + 100.10^3 - 25.10^3 = 25.000$

b) First of all, the mean profit function is $\dfrac{B(x)}{x} = -0,05x + 100 - \dfrac{25.000}{x}$
As this function is concave ($(\dfrac{B(x)}{x})" = \dfrac{-5.000}{x^3} < 0$), the critical point, if it exists, will be the unique absolute maximizer.
So, $(\dfrac{B(x)}{x})' = -0,05 + \dfrac{25.000}{x^2} = 0 \Longleftrightarrow$
$\Longleftrightarrow x^2 = \dfrac{25.000}{0,05} = \dfrac{2.500.000}{5} = 50.10^4 \Longleftrightarrow x_0 = 500\sqrt{2} \approx 700$
from that you can deduce that such level of production is the one which maximizes the mean profit.
Finally, what happens on the interval $[x_0, x_1] = [700, 1000]$? The following:
i) the profits keep on rising, as $B'(x) > 0$. Nevertheless,
ii) the mean profits decrease, as $(\dfrac{B(x)}{x})' < 0.$

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(4) **Let** $f(x) = \begin{cases} ax+1 & si\ x \leq -1 \\ x^2+b & si\ \ x > -1 \end{cases}$ **and consider** f **restricted to the interval** $[-2,3]$ **. We ask you to:**

(a) Determine a and b in order that $f(x)$ satisfies the hypothesis (or initial conditions) of the mean value (or Lagrange's) theorem in that interval.

(b) Let us suppose that $2a+b=-2, a \neq -2$. Determine, if they exist, the value or values of c in order that the thesis (or conclusion) of this theorem is satisfied.

Hint for both parts: write down the mean value (or Lagrange's) theorem.

1 point

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a) By Lagrange's theorem, it is required that the function is continuous on $[-2,3]$ and differentiable on $(2,3)$.

Obviously, the only point to study is $x=-1$. So:

i) $f(x)$ is continuous at $x=-1 \iff -a+1 = 1+b$.

ii) $f(x)$ is differentiable at $x=-1 \iff f(x)$ is continuous at that point and $a=-2$.

Then, $f(x)$ satisfies the hypothesis of Lagrange's theorem when $a=-2, b=2$.

b) As $a \neq -2$, the hypothesis of the theorem are not satisfied. Nevertheless, it can be the case that the thesis may be true.

In this case, as $f(3)-f(-2) = 9+b-(-2a+1) = 8+2a+b = 6$, because

$2a+b=-2$, the thesis of Lagrange's theorem claims that there exists c in the interval

$(-2,3)$ in such a way that:

$f(3)-f(-2) = 6 = f\,'(c)(3-(-2)) \iff f\,'(c) = \frac{6}{5}$; and, as the first derivative, although it doesn't exist at the point $x=-1$, satisfies that:

$$f'(x) = \begin{cases} a & si\ x < -1 \\ 2x & si\ \ x > -1 \end{cases}$$

you have two possible cases:

i) $a \neq \frac{6}{5} \Longrightarrow$ the only point c that satisfies the thesis will be $x > -1$ such that $2x = \frac{6}{5} \iff x = \frac{3}{5}$.

ii) $a = \frac{6}{5} \Longrightarrow$ the points c that satisfy the thesis will be the points of the set $(-2,-1) \cup \{\frac{3}{5}\}$.

Remark.

Lagrange's theorem: Let $g:[a,b] \longrightarrow \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b).

Then, there exists a point c in the interval (a,b) such that: $g(b)-g(a) = g'(c).(b-a)$

5. **Let** A **be the set between the graphs of the functions** $f(x) = -x^2+3x+4$ **and** $g(x) = x^2-x-2$**. We ask you to:**

   (a) Draw the set A and obtain the maximum, the minimum, the maximal and minimal elements of the set A, if they exist, using the Pareto order.

   (b) Compute the area of the region given by the set A.

   Hint: the Pareto order is given by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

   1 point

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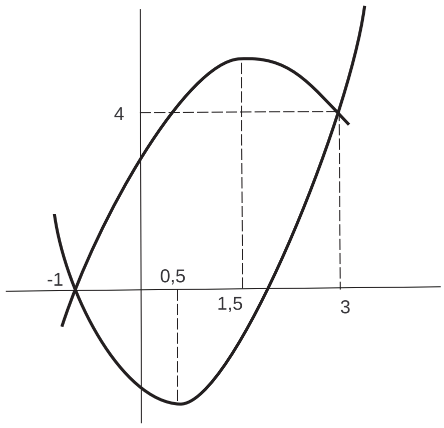
a) As $f(x) = g(x)$ is equivalent to $x = -1, x = 3$, $f(x)$ is concave and $g(x)$ is convex, the region is limited above by the function $f(x)$ and below by the function $g(x)$, functions that cross at the points $(-1, 0)$ and $(3, 4)$.
As $f'(x) = -2x+3 > 0 \iff x < \frac{3}{2}$, this inequality means that the function $f(x)$ is increasing on the interval $[-1, \frac{3}{2}]$ and decreasing on $[\frac{3}{2}, 3]$. Analogously, as the function $g'(x) = 2x-1 > 0 \iff x > \frac{1}{2}$, So, the region has a shape like this:



Obviously, maximum(A) doesn't exist, because
$\{$maximal points$(A)\} = \{(x, f(x)) : \frac{3}{2} \leq x \leq 3\}$.
Analogously, minimum(A) doesn't exist, because
$\{$minimal points$(A)\} = \{(x, g(x)) : -1 \leq x \leq \frac{1}{2}\}$.

b) The area asked is:
$\int_{-1}^{3} (f(x) - g(x))dx = \int_{-1}^{3} (-2x^2 + 4x + 6)dx = [-\frac{2}{3}x^3 + 2x^2 + 6x]_{-1}^{3} = 18 - (\frac{2}{3} + 2 - 6) =$
$= 21 + \frac{1}{3} = \frac{64}{3}$ area untis.

6. **Given the function** $f(x) = \dfrac{x-3}{x^2-3x+2}$, **we ask you to:**

(a) Compute, if it exists, the primitive of this function in the interval $(-\infty, 1)$ satisfying $F(0) = 0$.

(b) Let us consider the interval $(2, \infty)$ and the primitive $F(x)$ of $f(x)$ satisying $F(3) = 0$.Draw the graph of $F(x)$.

Hint for part b): it is enough to find the intervals where $F(x)$ increases or decreases, global extrema and the limits of $F(x)$ in 2^+ and ∞. It is not necessary to study the existence of asymptotes in ∞.

1 point

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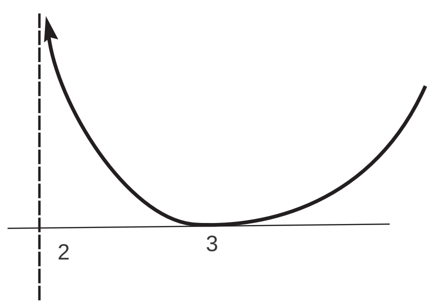
a) As $f(x)$ is a rational function, we compute its indefinite integral by the method of simple fractions. So, as $x^2 - 3x + 2 = (x-1)(x-2)$, $f(x) = \dfrac{x-3}{x^2-3x+2} = \dfrac{A}{x-1} + \dfrac{B}{x-2} \iff$
$\iff x - 3 = A(x-2) + B(x-1)$. And now, if $x = 1 \Longrightarrow A = 2$; and if $x = 2 \Longrightarrow B = -1$.
So, in any of those intervals, $F(x) = \int f(x)dx = 2\ln|x-1| - \ln|x-2| + C$.
In particular, on the interval $(-\infty, 1)$,
$F(0) = -\ln 2 + C = 0 \Longrightarrow F(x) = 2\ln|x-1| - \ln|x-2| + \ln 2$.

b) By the fundamental calculus theorem, it holds that $F'(x) = f(x)$. For that reason, it is satisfied that:
i) $F(x)$ is increasing on the interval $[3, \infty)$, because on that interval $F'(x) = f(x) > 0$.
ii) $F(x)$ is decreasing on the interval $(2, 3]$, because on that interval $F'(x) = f(x) < 0$.
So $F(x)$ reaches an absolute minimum at $x = 3$.
On the other hand, as $F(x) = 2\ln|x-1| - \ln|x-2| + C$, it holds that $\lim\limits_{x \to 2+} F(x) = \infty$.
Finally, observing that $F(x) = \ln[\dfrac{(x-1)^2}{x-2}] + C$, it holds that
$\lim\limits_{x\to\infty} F(x) = \ln\infty + C = \infty$.
Since $F(3) = 0$ the graph of $F(x)$ will look like this:



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