Universidad Carlos III de Madrid

Question	1	2	3	4	5	6	Total
Grade							

Economics Department **Final Exam, Mathematics I** **January 13, 2012**

Total time: 2 hours.

LAST NAMES:	**FIRST NAME:**
DNI: **Title:**	**Group:**

(1) **Let** $f(x) = \dfrac{e^x}{e^x - 1}$ **. About function** f**:**

(a) Find its domain, its increasing and decreasing intervals, and find its maxima and minima, both local and global.

(b) Find all its asymptotes, state its range (image), and sketch its graph.

1 point

---

a) The domain of f is all the real line, except for point $x = 0$, where its denominator vanishes.
More formally, Domain$(f) = (-\infty, 0) \cup (0, \infty)$.
On the other hand, since $f\,'(x) = \dfrac{e^x(e^x - 1) - e^x.e^x}{(e^x - 1)^2} = \dfrac{-e^x}{(e^x - 1)^2} < 0$,
we have that f is decreasing at intervals $(-\infty, 0)$ and $(0, \infty)$.
Hence, it is never increasing, and it does not have maxima or minima, be it local or global.

b) In order to find the vertical asymptotes of f, it is sufficient to take into account that

$$\lim_{x \longrightarrow 0^-} f(x) = \lim_{x \longrightarrow 0^-} \frac{e^x}{e^x - 1} = \tfrac{1}{0^-} = -\infty$$

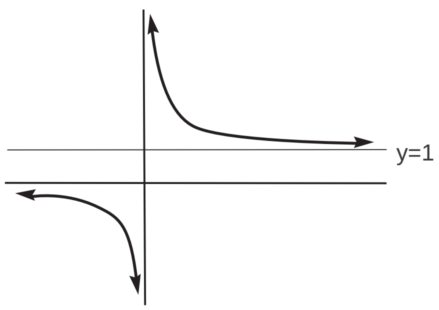
$$\lim_{x \longrightarrow 0^+} f(x) = \lim_{x \longrightarrow 0^+} \frac{e^x}{e^x - 1} = \tfrac{1}{0^+} = \infty$$

Hence, f has vertical asymptotes at $x = 0$. On the other hand,

$$\lim_{x \longrightarrow -\infty} f(x) = \lim_{x \longrightarrow -\infty} \frac{e^x}{e^x - 1} = \tfrac{0}{-1} = 0; \text{ and}$$

$$\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} \frac{e^x}{e^x - 1} = \tfrac{\infty}{\infty} = (L'Hopital) = \lim_{x \longrightarrow \infty} \frac{e^x}{e^x} = 1.$$

So that f has horizontal asymptotes $y = 0$ at $-\infty$ and $y = 1$ at $+\infty$.
Taking this into account, and the fact that f is always decreasing and continuous in its domain, the range, or image, is:
Im(f)$= (-\infty, 0) \cup (1, \infty)$.
The sketch of the graph of f is :



(2) **Let** $y = f(x)$ **be a function defined implicitly by equation** $e^{x+y} + x^2y = e$, **in a neighborhood of** $x = 0, y = 1$.

(a) Find the equation of the tangent line to the graph of f at the point $x = 0, y = 1$.

(b) Find $f"(0)$ and sketch the graph of f around point $x = 0, y = 1$.

Hint: In order to sketch the graph of f you only need part (a), and use the fact that $f"(0) < 0$.

1 point

---

a) First of all, we derive the equation that defines the implicit function:

$e^{x+y}(1 + y') + 2xy + x^2y' = 0$

Next, we substitute at $x = 0, y = 1$, and get that

$e(1 + y') = 0 \Longrightarrow y' = -1$

It follows that the equation of the tangent line is: $y - 1 = -(x - 0)$, or $x + y = 1$.

b) Deriving again, for the second time, the equation that defines the implicit function, we get that:

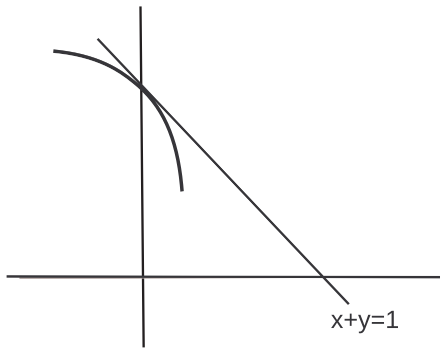
$e^{x+y}(1 + y')^2 + e^{x+y}y" + 2y + 2xy' + 2xy' + x^2y" = 0$

Next, we substitute at $x = 0, y = 1, y' = -1$ and get that

$ey" + 2 = 0 \Longrightarrow y" = \dfrac{-2}{e} < 0$

It follows that f is concave at $x = 0$.

Hence, the graph of f next to $x = 0$, would be like:



(3) **Let** $C(x) = C_0 + 10x + 0,03x^2$ **and** $p(x) = 50 - 0,01x$ **be the cost and demand functions, respectively, of a monopolistic firm. Using these functions,**

(a) Find the production level x_0 at which the firm maximizes its benefit.

(b) Find fixed cost C_0 such that the otput at which average cost (or medium cost) is minimized is $x = 200$.

Remark: justify your answers.

1 point

---

a) The income function is $I(x) = 50x - 0,01x^2$, so that the benefit function is $B(x) = I(x) - C(x) = -0,01x^2 + 50x - (0,03x^2 + 10x + C_0) = -0,04x^2 + 40x - C_0$.
This function is concave, since $B"(x) < 0$.
Hence, the critical point, if it exists, would be the unique global maximum.
Since $B'(x) = -0,08x + 40$; $B'(x) = 0 \iff x = \dfrac{40}{0,08} = 500$, which is the production level that maximizes benefits.

b) First of all, the average cost function is $\dfrac{C(x)}{x} = \dfrac{C_0}{x} + 10 + 0,03x$.
Since this function is convex $((\dfrac{C(x)}{x})" > 0$), the critical point, if it exists, would be the unique global minimum.
We have $(\dfrac{C(x)}{x})' = -\dfrac{C_0}{x^2} + 0,03$; $(\dfrac{C(x)}{x})' = 0 \iff x^2 = \dfrac{C_0}{0,03} \iff C_0 = 0,03.200^2 = 1.200$, which is the fixed cost such that average costs are minimized when $x = 200$.

ANNEX: SOLUTIONS FOR PROBLEMS 1, 2 AND 3

4. **Let** $0 < a < 1$**, and consider function** $f : [a, \dfrac{1}{a}] \to \mathbb{R}$**, defined by** $f(x) = \dfrac{1}{x}$ **.**

   (a) State the mean (or medium) value Theorem (Lagrange's Theorem) for general conditions.

   (b) Determine the value of c in a way that it fulfills the thesis (or conclusion) of that Theorem for our function f.

   1 point

---

a) Let $g : [a, b] \longrightarrow \mathbb{R}$ be continuous on $[a, b]$ and derivable on (a, b).
Then, there exists a point c in the interval (a, b) such that
$g(b) - g(a) = g'(c).(b - a)$

b) If we let $g(x) = f(x) = \frac{1}{x}, a = a, b = \frac{1}{a}$, we have that there is a c in $(a, \frac{1}{a})$ such that
$a - \dfrac{1}{a} = -\dfrac{1}{c^2}(\dfrac{1}{a} - a) \Longleftrightarrow \dfrac{1}{c^2} = 1$. Since c belongs to $(a, \frac{1}{a})$, it should be positive, and hence $c = 1$.

5. **Let** $A = \{(x, y) \in \mathbb{R}^2 : \frac{4}{9}x^2 \leq y \leq \frac{4}{3}\sqrt{3x}\}$**.**

(a) Represent the set A and find its maximal and minimal sets, as well as its maximum and minimum points, if they exist.

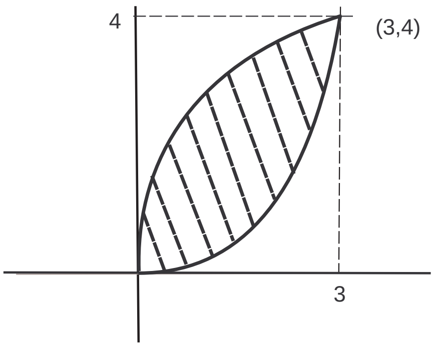
(b) Find the area of A.

Hint: the Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$.

1 point

---

a) The set is a subset of the first quadrant, and is enclosed within curves $y = \frac{4}{9}x^2, y = \frac{4}{3}\sqrt{3x}$.
The points (x, y) where these curves intercept each other satisfy
$\frac{4}{9}x^2 = \frac{4}{3}\sqrt{3x} \Longrightarrow x^2 = 3\sqrt{3x} \Longrightarrow x^4 = 27x \Longrightarrow$
i) $x = 0 \Longrightarrow y = 0$, hence $(0, 0)$ is one of the intercepts. Also
ii) $x^3 = 27 \Longrightarrow x = 3 \Longrightarrow y = 4$, hence $(3, 4)$ is the remaining intercept.
We can now sketch the graph of A:



Obviously, $Maximum(A) = \{\text{Maximals}(A)\} = \{(3, 4)\}$.
$Minimum(A) = \{\text{Minimals}(A)\} = \{(0, 0)\}$.

b) The area within A is:
$\int_0^3 (\frac{4}{3}\sqrt{3x} - \frac{4}{9}x^2)dx = [\frac{4}{3}\sqrt{3}\frac{x^{3/2}}{3/2} - \frac{4}{9}\frac{x^3}{3}]_0^3 = 8 - 4 = 4$ area units.

6. **Let** $F(x) = \int_3^x f(t)dt$ **be defined for** $x \in [3,5]$, **where** $f : [3,5] \longrightarrow \mathbb{R}$ **is a strictly decreasing and continuous function, with** $f(3) = 1, f(4) = 0, f(5) = -1$.

(a) Find the intervals at which $F(x)$ is increasing or decreasing, and study the existence of its global maxima and minima.

(b) Estimate the value of $F(5) = \int_3^5 f(t)dt$.

Remark: In (a), justify all you can say about $F(x)$.

1 point

---

a) By the Fundamental Theorem of Calculus, $F'(x) = f(x)$. Hence

i) $F(x)$ is increasing in $[3,4]$, since on that interval $F'(x) = f(x) > 0$.

ii) $F(x)$ is decreasing in $[4,5]$, since on that interval $F'(x) = f(x) < 0$.

It follows that $F(x)$ reaches a global maximum at $x = 4$.

On the other hand, $F(x)$ reaches a global minumum at $x = 3$ or at $x = 5$, or at both points, depending on whether $F(3)$ is less than, bigger than, or equal to $F(5)$.

b) We have that $F(5) = \int_{\mathbf{3}}^{\mathbf{4}} \mathbf{f(t)dt} + \int_{\mathbf{4}}^{\mathbf{5}} \mathbf{f(t)dt}$. Now

$0 = 1 \cdot 0 \leq \int_{\mathbf{3}}^{\mathbf{4}} \mathbf{f(t)dt} \leq \mathbf{1 \cdot 1 = 1}$, since $0 \leq f(t) \leq 1$ when $3 \leq t \leq 4$; and

$-1 = 1 \cdot (-1) \leq \int_{\mathbf{4}}^{\mathbf{5}} \mathbf{f(t)dt} \leq \mathbf{1 \cdot 0 = 0}$, since $-1 \leq f(t) \leq 0$ when $4 \leq t \leq 5$.

Adding up both inequalities we get

$-1 \leq \mathbf{F(5)} \leq \mathbf{1}$

ANNEX: SOLUTIONS FOR PROBLEMS 4, 5 Y 6