Universidad Carlos III de Madrid

Question	1	2	3	4	5	6	Total
Grade							

Economics Department **Final Exam, Mathematics I** **January 14, 2011**

Total length: 2 hours.

SURNAME:		**NAME:**
DNI:	**Degree:**	**Group:**

1. The function f is defined by $f(x)=\begin{cases} 2-4(x-4)^2 & if\ 3\leq x\leq 4 \\ 2+4(x-4)^2 & if\ \ 4\leq x\leq 5 \end{cases}$ **.**

a) Find the range of f and find the inverse function f^{-1}.

b) Sketch the graph of the inverse function f^{-1}. Find its domain and its range.

Hint: sketch the graph of the function f and observe if it is an increasing or decreasing function.

1 point.

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a) The function f is continuous and strictly increasing on its domain. Finding the values of $f(3)=-2$ and $f(5)=6$ we know that the range of the function is $[-2,6]$.

In order to find the inverse function f^{-1}. firstly, we consider f defined on the subdomain $[3,4]$, whose range is the interval $[-2,2]$.

$y=2-4(x-4)^2 \iff (x-4)^2=\dfrac{2-y}{4} \iff x-4=-\sqrt{\dfrac{2-y}{4}} \iff x=4-\dfrac{\sqrt{2-y}}{2}$;

So, $f^{-1}(x)=4-\dfrac{\sqrt{2-x}}{2}$, whenever $x\in[-2,2]$.

Secondly, we consider f defined on the subdomain $[4,5]$, whose range is the interval $[2,6]$.

$y=2+4(x-4)^2 \iff (x-4)^2=\dfrac{y-2}{4} \iff x-4=\sqrt{\dfrac{y-2}{4}} \iff x=4+\dfrac{\sqrt{y-2}}{2}$;

So, $f^{-1}(x)=4+\dfrac{\sqrt{x-2}}{2}$, whenever $x\in[2,6]$.

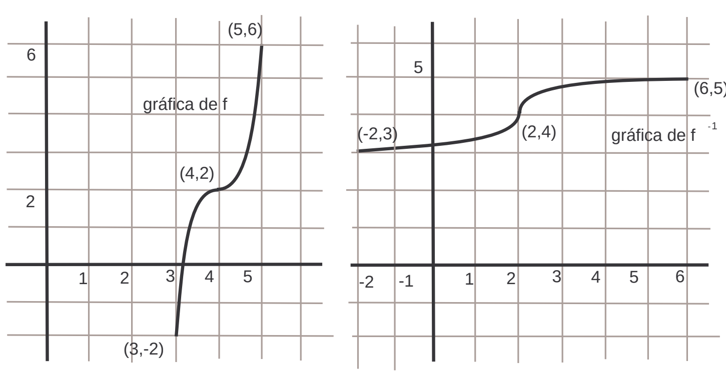
Thus we have: $f^{-1}(x)=\begin{cases} 4-\dfrac{\sqrt{2-x}}{2} & si\ \ -2\leq x\leq 2 \\ 4+\dfrac{\sqrt{x-2}}{2} & si\ \ 2\leq x\leq 6 \end{cases}$

b) To sketch the graph of the inverse function f^{-1}, it is useful to bear in mind that as f is strictly increasing it will also be its inverse function.

We also know that for the inverse function f^{-1}, the domain is $[-2,6]$, the range is $[3,5]$ and $f(2)=4$.

And furthermore, since f is concave on the interval $[3,4]$ and convex on the interval $[4,5]$, thus f^{-1} is convex on $[-2,2]$ and concave on $[2,6]$.

Thus, the graph of f^{-1} will approximatly be:



2. The function f is defined by

$$f(x) = \begin{cases} \dfrac{x^3}{1+x^2}, & \text{if } x<0 \\ a, & \text{if } x=0 \\ \dfrac{\ln(x^2+1)}{x}, & \text{if } x>0 \end{cases}$$

where $a \in \mathbb{R}$, is a real number.

a) Find the value of a giving your justification, if there is any, such that the function f is continuous and/or differentiable at $x=0$.

b) Find the vertical, horizontal and oblique asymptotes of the function f for each value of a.

1 point.

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a) Firstly, in order to study whether the function is continuous at the point $x=0$, we calculate the sided limits of $f(x)$ at $x=0$.

$\lim\limits_{x\longrightarrow 0^-} f(x) = 0$, $f(0)=a$, $\lim\limits_{x\longrightarrow 0^+} f(x) = \frac{0}{0} = (L'Hopital) = \lim\limits_{x\longrightarrow 0^+} \dfrac{2x}{x^2+1} = 0$;

In this case, f is continuous at $x=0$ when $a=0$.

Now, if we suppose that $f(x)$ is continuous at $x=0$, $f(x)$ is differentiable at $x=0$ when $\lim\limits_{x\longrightarrow 0^-} f'(x) = \lim\limits_{x\longrightarrow 0^+} f'(x)$. But:

$$\lim_{x\longrightarrow 0^-} f'(x) = \lim_{x\longrightarrow 0^-} \frac{3x^2(1+x^2)-x^3 2x}{(1+x^2)^2} = 0.$$

and, $\lim\limits_{x\longrightarrow 0^+} f'(x) = \lim\limits_{x\longrightarrow 0^+} \dfrac{[2x/(x^2+1)]x - \ln(x^2+1)}{x^2} = \lim\limits_{x\longrightarrow 0^+} \dfrac{[2x/(x^2+1)]x}{x^2} - \lim\limits_{x\longrightarrow 0^+} \dfrac{\ln(x^2+1)}{x^2}$;

Calculating the value of both limits apart:

$\lim\limits_{x\longrightarrow 0^+} \dfrac{[2x/(x^2+1)]x}{x^2} = \lim\limits_{x\longrightarrow 0^+} \dfrac{2}{x^2+1} = 2$. And the other one,

$\lim\limits_{x\longrightarrow 0^+} \dfrac{\ln(x^2+1)}{x^2} = \frac{0}{0} = (L'Hopital) = \lim\limits_{x\longrightarrow 0^+} \dfrac{2x/(x^2+1)}{2x} = 1.$

We finally obtain $\lim\limits_{x\longrightarrow 0^+} f'(x) = 1$.

Thus, we can assert that $f(x)$, in no case is differentiable at $x=0$.

b) Obviously, vertical asymptotes do not exist. To find asymptotes at infinity:

$$\lim_{x\longrightarrow -\infty} \frac{f(x)}{x} = \lim_{x\longrightarrow -\infty} \frac{x^3}{(1+x^2)x} = 1,$$

$$\lim_{x\longrightarrow -\infty} f(x) - x = \lim_{x\longrightarrow -\infty} \frac{x^3}{1+x^2} - \frac{x(1+x^2)}{1+x^2} = \lim_{x\longrightarrow -\infty} \frac{-x}{1+x^2} = 0$$

Then f has an oblique asymptote $y=x$ at $-\infty$.

$$\lim_{x\longrightarrow \infty} f(x) = \lim_{x\longrightarrow \infty} \frac{\ln(x^2+1)}{x} = \frac{\infty}{\infty} = \lim_{x\longrightarrow \infty} \frac{2x}{x^2+1} = 0$$

Thus the function has a horizontal asymptote $y=0$ at ∞.

3. Consider the equation $x + e^{4x} = b$.

a) Prove that there is always only one solution of the given equation.

b) Find out, when $b = 0$, the solution of the equation with an absolute error less than 0.25.
Hint: the value of e falls in between 2 and 3.
1 point.

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a) The function $f(x) = x + e^{4x}$ is strictly increasing on $\mathbb{R}$, from its first derivative $f'(x) = 1 + 4e^{4x} > 0$. Therefore, if there is any solution, there is only one.
In order to prove that there is a solution of the equation we should notice that the function is continuous on its domain and from
$\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$, (range $(f) = \mathbb{R}$)
we can state that the solution always exits for each $b \in \mathbb{R}$ and it is unique.

b) First, we try with $f(0) = 1 > 0$. So since the function is strictly increasing, we know the root is to the left of (i.e. smaller than) $x = 0$.
Since $f(-1) = -1 + e^{-4} = -1 + \dfrac{1}{e^4} < -1 + \frac{1}{16} < 0$, using Bolzano's Theorem the root is in the interval $(-1, 0)$.
Finally, if we try the midpoint of the above interval, since $f(-\frac{1}{2}) = -\frac{1}{2} + \dfrac{1}{e^2} < -\frac{1}{2} + \frac{1}{4} < 0$. then we know for the same reason as above, the root is in the interval $(-\frac{1}{2}, 0)$, and in this case we can take the midpoint $x = -\frac{1}{4}$ as an approximate value of the root with an absolute error of less than 0.25.

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4. The function f is defined by $f(x)=\begin{cases} x^2-1 & if\ x<0 \\ 1-2x & if\ \ 0\le x \end{cases}$, and we consider the function $f\colon [a,b] \to \mathbb{R}$, where $a<b$ are real numbers.

a) Write down Weierstrass' Theorem and find the values of a and b such that the hypothesis (or initial condition) is true in the theorem.

b) Find the values of a and b such that the hypothesis is NOT satisfied but the thesis (or conclusion) is.

Hint: Draw the graph of the function.

1 point.

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a) The hypothesis (conditions) is satisfied when the function is continuous. Therefore this happens when $0\le a$ or when $b<0$.

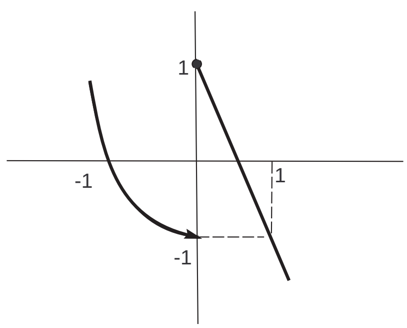
b) On the one hand the hypothesis is not satisfied when the function f is discontinuous, this is the case when $a<0\le b$.

On the other hand, for each value of a and b the function always attains its global maximum.

In order to determine the solution, we just need to consider the case that f is discontinuous and attains its global minimum. Since the function is only left handed discontinuous at $x=0$ and since $\lim_{x\to 0^-} f(x) = -1 < f(0)$, and we notice that $f(x)\le -1 \iff x\ge 1$

then, $f(x)$ is discontinuous and satisfies the thesis when $a<0<1\le b$.

Look at the graph of f in order to fully understand the situation.



5. Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 - 2x + 1 \leq y \leq -x^2 + 2x + 1\}$ **be a set of points.**

a) Draw the set A and obtain the maximum, the minimum, the maximal elements and minimal elements of the set A if they exist.

b) Calculate the area of the region given by the set A.

Hint: The Pareto ordering is: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

1 point.

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a) $f(x) = x^2 - 2x + 1$ describes a convex parabola whose vertex is the point $(1, 0)$, since $f\,'(1) = 0,\ f(1) = 0, f"(1) > 0$.

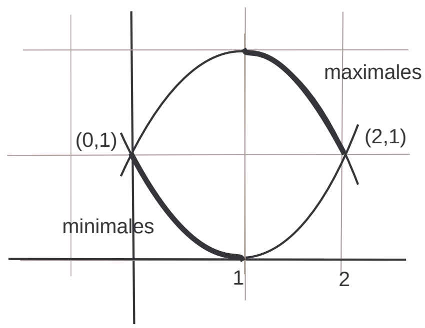
$g(x) = -x^2 + 2x + 1$ describes a concave parabola whose vertex is the point $(1, 2)$, since $g\,'(1) = 0,\ g(1) = 2, g"(1) < 0$.

To find the points (x, y) of intersection on both parabolas we solve the equation:

$x^2 - 2x + 1 = -x^2 + 2x + 1 \iff 2x^2 = 4x \iff x = 0, x = 2.$

to obtain: $(0, 1), (2, 1)$.

Thus the region bound by set A is:



It is certain that there are neither $\max(A)$ nor $\min(A)$, since

maximal points of $(A) = \{(x, y) : y = -x^2 + 2x + 1, 1 \leq x \leq 2\}$

minimal points of $(A) = \{(x, y) : y = x^2 - 2x + 1, 0 \leq x \leq 1\}$.

b) The area of the region is:

$$\int_0^2 [(-x^2 + 2x + 1) - (x^2 - 2x + 1)]dx = \int_0^2 (-2x^2 + 4x)dx = \left[\frac{-2x^3}{3} + \frac{4x^2}{2}\right]_0^2 = \frac{-16}{3} + 8 =$$

$= \dfrac{8}{3}$ area units.

6. Given the function $F(x) = \int_{-2}^{x} t^3 e^{-t^2} dt$**, defined at** $x \in [-2, 2]$.

a) Find the intervals in which $F(x)$ is increasing/decreasing. Find the local and/or global maxima and minima of $F(x)$.

b) Find the intervals in which $F(x)$ is convex/concave. Locate any inflection points of $F(x)$. Notice: It is neither necessary nor helpful to find the primitive function of $f(x) = x^3 e^{-x^2}$.

1 point.

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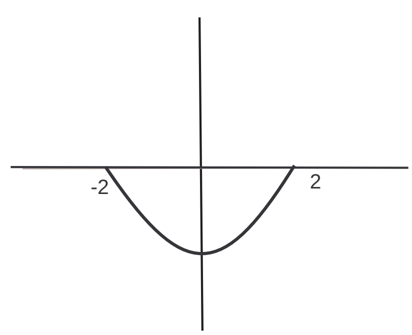
a) Referring to the Fundamental Theorem of Calculus we know that $F'(x) = x^3 e^{-x^2}$. Therefore, the following is satisfied:
$F'(x) < 0 \iff x < 0; F'(x) > 0 \iff x > 0.$
So we know that $F(x)$ is strictly decreasing on the interval $(-2, 0)$ and strictly increasing on the interval $(0, 2)$.
Firstly we can conclude that $F(x)$ attains its global (and local) minimum at $x = 0$.
And secondly, to find the maximum of the function $F(x)$, since $f(x) = x^3 e^{-x^2}$ is an odd function, then
$\int_{-2}^{0} t^3 e^{-t^2} dt = -\int_{0}^{2} t^3 e^{-t^2} dt$ and we know that $F(-2) = 0 = F(2)$.
We can conclude that $F(x)$ attains its global maximum at the points $x = -2, x = 2$.

b) Since $F"(x) = f\ '(x) = x^2(-2x^2 + 3)e^{-x^2}$, then:
$F"(x) = 0 \iff x = 0, x = \pm\sqrt{\frac{3}{2}}$; Furthermore, since
$F"(x) < 0 \iff x \in (-2, -\sqrt{\frac{3}{2}}) \cup (\sqrt{\frac{3}{2}}, 2)$; (these are two different open intervals where the function is concave) and
$F"(x) > 0 \iff x \in (-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}})$; (this is an open interval where the function is convex).
We can thus state that the function has two inflection points at $x = \pm\sqrt{\frac{3}{2}}$.
A sketch of the graph of F is:



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