Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	Total
Points						

Department of Economics **Mathematics I Final Exam** **January 11th 2023**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = (x^2-4)^{\frac{2}{3}}$, defined in the interval $[0,\infty)$. Then:**

(a) find the intervals where $f(x)$ increases and decreases, its global maximum and minimum, and range (or image) of $f(x)$.

(b) find the intervals where $f(x)$ is convex and concave, and its points of inflection. Draw the graph of the function.

(c) consider $f_b(x)$ to be the function $f(x)$ defined on the interval $[0,b]$, where $b \geqslant 2$. Find the global maximum (and the global maximizers) of $f_b(x)$.

0.4 points part a); 0.4 points part b); 0.2 points part c)

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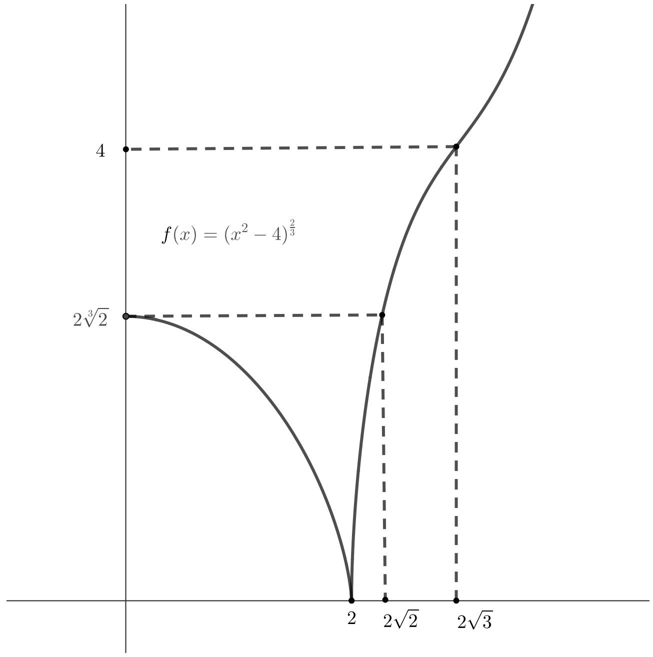
a) $f(x)$ is continuous on its domain, $[0,\infty)$. Since $y = x^{\frac{2}{3}}$ is derivable everywhere but at $x=0$, $f(x)$ is also derivable everywhere but when $x^2-4=0$, that is, at $x=2$.
Since $f'(x) = \dfrac{2\cdot 2x}{3(x^2-4)^{\frac{1}{3}}}$, the critical points are $x=0$ and $x=2$.
$f'(1)<0$, then $f'(x)<0$ if $x\in(0,2)$, so, $f(x)$ is decreasing on $[0,2]$.
$f'(3)>0$, then $f'(x)>0$ if $x\in(2,\infty)$, so, $f(x)$ is increasing on $[2,\infty)$.
Obviously, $x=2$ is the global minimizer of $f(x)$ since $f(2)=0<f(x)$ if $x\neq 2$.
Moreover, $f(x)$ has no global maximum, since $\lim\limits_{x\longrightarrow\infty} f(x)=\infty$.
Finally, it is deduced that the range of $f(x)$ is $[0,\infty)$.

b) There exist $f''(x)$ for any $x\neq 2$. And since,
$$f''(x) = \frac{4}{3}\frac{(x^2-4)^{\frac{1}{3}} - x\cdot(\frac{1}{3})(x^2-4)^{-\frac{2}{3}}\cdot 2x}{(x^2-4)^{\frac{2}{3}}} =$$
[multiplying numerator and denominator by $3(x^2-4)^{\frac{2}{3}}$]
$$= \frac{4}{9}\cdot\frac{3(x^2-4)-2x^2}{(x^2-4)^{\frac{4}{3}}} = \frac{4}{9}\frac{x^2-12}{(x^2-4)^{\frac{4}{3}}}$$
and the second order derivative is equal to zero at $\sqrt{12}=2\sqrt{3}$.
$f''(1)<0$, then $f''(x)<0$ if $x\in(0,2)$, so, $f(x)$ is concave on $[0,2]$.
$f''(3)<0$, then $f''(x)<0$ if $x\in(2,2\sqrt{3})$, so, $f(x)$ is concave on $[2,2\sqrt{3}]$.
$f''(4)>0$, then $f''(x)>0$ if $x\in(2\sqrt{3},\infty)$, so, $f\ (x)$ is convex on $[2\sqrt{3},\infty)$.
Therefore, it is deduced that $\sqrt{12}=2\sqrt{3}$ is a point of inflection.
Notice: $x=2$ is not an inflection point and $f(x)$ is not concave on $[0,2\sqrt{3}]$ either, since, the line segment that joints the points $(1,f(1))$ and $(3,f(3))$ is not underneath the graph of $f(x)$ at the point $x=2$.
The graph of f will have an appearance approximately, similar to the one in the figure at the end.

c) We know that $f(x)$ is decreasing on $[0,2]$ and increasing on $[2,\infty)$.
Therefore, naming x^* to the unique number in the interval $(2,\infty)$ that satisfies:
$f(x^*) = f(0) = 4^{\frac{2}{3}}$, then:
if $b<x^* \Longrightarrow Max(f_b) = f(0) = 4^{\frac{2}{3}}$; maximizer $(f_b)=\{0\}$.
if $b=x^* \Longrightarrow Max(f_b) = f(0) = 4^{\frac{2}{3}}$; maximizer $(f_b)=\{0,x^*\}=\{0,b\}$.
if $b>x^* \Longrightarrow Max(f_b) = f(b) = (b^2-4)^{\frac{2}{3}}$; maximizer $(f_b)=b$.
¿What is the value of x^*? Since $f(x^*) = (x^{*2}-4)^{\frac{2}{3}} = 4^{\frac{2}{3}} \Longrightarrow x^{*2}-4=4 \Longrightarrow x^{*2}=8 \Longrightarrow x^*=2\sqrt{2}$
Look again at the draw of the graph!



4
$f(x) = (x^2 - 4)^{\frac{2}{3}}$
$2\sqrt[3]{2}$
2
$2\sqrt{2}$
$2\sqrt{3}$

(2) **Given the implicit function $y = f(x)$, defined by the equation $4x^2 + 2y - y^3 = 1$ in a neighbourhood of the point $x = 0, y = 1$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 0$.

(b) sketch the graph of the function f near the point $x = 0$.

(c) consider $f_\delta(x)$ the implicit function defined in the interval $[0, \delta)$. Sketch the graph of its inverse function.

Using Taylor Polynomial, find the approximate formula of $f_\delta(x)$ **inverse function**.

(*Hint for part (b) and (c):* use $f''(0) > 0$).

0.4 points part a); 0.2 points part b); 0.4 points part c).

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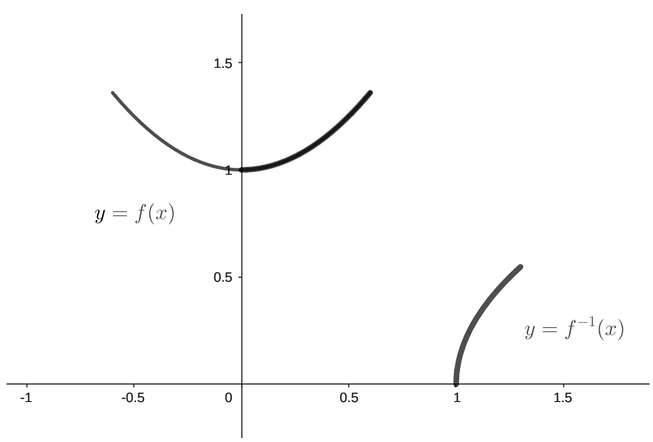
a) First of all, we notice that the point $(0, 1)$ is a solution of the equation.
Secondly, we calculate the first-order derivative of the equation:
$8x + 2y' - 3y^2y' = 0$
evaluating at $x = 0,\ y(0) = 1$ we obtain: $y'(0) = f'(0) = 0$.
Then, the equation of the tangent line is: $y = P_1(x) = 1$.
Analogously, we calculate the second-order derivative of the equation:
$8 + 2y'' - 3y^2y'' - 6y(y')^2 = 0$
evaluating at $x = 0$, $y(0) = 1$, $y'(0) = 0$ we obtain: $y''(0) = f''(0) = 8$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 1 + 4x^2$.

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be as you can see in the figure underneath.

c) The graph of $f_\delta(x)$ inverse function is simmetric with respect to the main diagonal $(y = x)$, then it will be represented as you can see in the same figure aunderneath.
Moreover, using second order Taylor Polynomial, we know that for $x \approx 0$:
$f_\delta(x) \approx 1 + 4x^2$, so the inverse function of Taylor Polynomial, for values of $x \geqslant 0$, will be given by the equation: $1 + 4y^2 = x \Longrightarrow y^2 = (x-1)/4 \Longrightarrow y = \frac{1}{2}\sqrt{x-1}$
Then, $f_\delta^{-1}(x) \approx \frac{1}{2}\sqrt{x-1}$, for $x \approx 1, x \geq 1$.



(3) **Let $C(x) = C_0 + 2x + x^2$ be the cost function and $p(x) = a - 5x$ the inverse demand function of a monopolistic firm, with $a, C_0 > 0$, $x \geqslant 0$. Then:**

(a) calculate the value of the parameter a, knowing that the production level to maximize the profit is $x^* = 4$.

(b) calculate the value of the parameter C_0, knowing that the production level to minimize the average cost is $x^* = 4$.

(c) state Rolle's Theorem. For the profit function of part (a), find the intervals $[\alpha, \beta]$ where:

i) the hypotheses or conditions of the theorem are satisfied.

ii) the thesis or result of the theorem is verified. Notice that in this case not every condition of the hypothesis needs to be satisfied.

0.4 points part a); 0.3 points part b); 0.3 points part c).

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a) First of all, we calculate the profit function.
$B(x) = (a - 5x)x - (C_0 + 2x + x^2) = -6x^2 + (a - 2)x - C_0$
Secondly, we calculate the first and second order derivatives of B:
$B'(x) = -12x + a - 2$; $B''(x) = -12 < 0$
we see that B has a unique critical point at $x^* = \dfrac{a-2}{12}$ and, since B is a concave function, the critical point is the unique global maximizer.

Finally, $x^* = 4 = \dfrac{a-2}{12} \Longrightarrow a = 50$.

b) The average cost function is $\dfrac{C(x)}{x} = \dfrac{C_0}{x} + 2 + x$,
its first order derivative: $\left(\dfrac{C(x)}{x}\right)' = -\dfrac{C_0}{x^2} + 1 = 0 \Longleftrightarrow x^2 = C_0$.
Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{2C_0}{x^3} > 0$, the function is convex and the critical point will be the global minimizer.
Then $x^* = 4 \Longrightarrow C_0 = 16$.

c) The hypotheses are that $B(x)$ must be continuous in the interval $[\alpha, \beta]$, derivable in the interval (α, β) and $B(\alpha) = B(\beta)$.
Since $B(x)$ is a parabola, its graph is symmetric with respect to the line $x = 4$, so $0 \leq \alpha < \beta$ must satisfied $(\alpha + \beta)/2 = 4 \Longrightarrow \beta = 8 - \alpha$, $\alpha \in [0, 4)$.
The thesis is that exist $\gamma \in (\alpha, \beta)$ such that $B'(\gamma) = 0$.
Obviously, it is satisfied if $0 \leq \alpha < 4 < \beta$, since $B'(4) = 0$.

(4) **Let** $f(x) = \left\{\begin{array}{ll} \sqrt{3+e^{2x}} & , x \leq 0 \\ \sqrt{a-be^{-x}} & x > 0 \end{array}\right\}$, **be a piece-wise defined function in** $\mathbb{R}$, where $a > b > 0$.

Then:

(a) Calculate a and b such that $f(x)$ is derivable at $x = 0$.

(b) for the function $f(x)$ study the existence of an asymptote at $-\infty$ and find its intervals of convexity and concavity on $(-\infty, 0)$.

(c) find the intervals where $f(x)$ increases and decreases and draw the graph of the function on $(-\infty, 0]$ (first piece).

0.4 points part a); 0.3 points part b); 0.3 points part c)

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a) First of all, we need the function f to be continuous at $x = 0$.
Since $\lim_{x \longrightarrow 0+} f(x) = \sqrt{a-b}$, and $f(0) = 2 = \lim_{x \longrightarrow 0^-} f(x)$, we obtain that the function is continuous on $[-1, 1]$ if $a - b = 4$.
Moreover, supposing f continuous, the function will be derivable at $x = 0$ when:
$\lim_{x \to 0+} f'(x) = f'(0^+)$ is equal to $f'(0^-)$. So, we obtain:

i) $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \dfrac{be^{-x}}{2\sqrt{a-be^{-x}}} = \dfrac{b}{2\sqrt{a-b}} = \dfrac{b}{4}$;

ii) $x < 0 \Longrightarrow f'(x) = \dfrac{2e^{2x}}{2\sqrt{3+e^{2x}}} \Longrightarrow f'(0^-) = \dfrac{2}{4}$.

Then, $f(x)$ is derivable at $x = 0$ if $b = 2$, $a = 6$.

b) $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \sqrt{3+e^{2x}} = \sqrt{3}$. Then $y = \sqrt{3}$ is the horizontal asymptote of the function at $-\infty$.

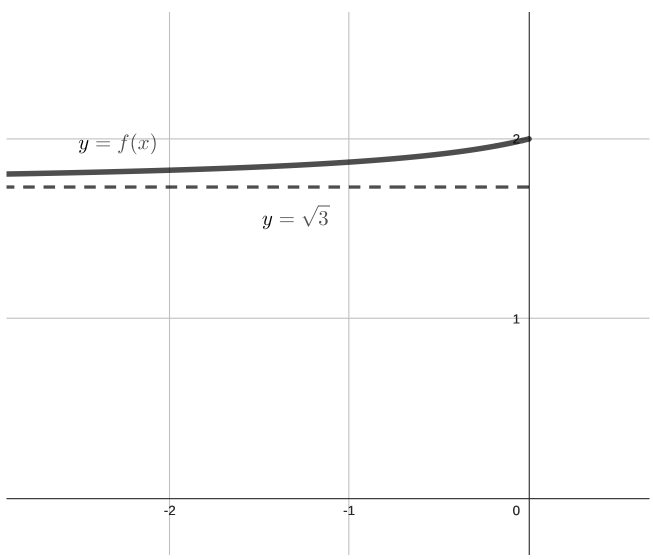
About convexity and concavity, we observe that:

$$x < 0 \Longrightarrow f''(x) = \frac{2e^{2x}\sqrt{3+e^{2x}} - e^{2x}(2e^{2x}/2\sqrt{3+e^{2x}})}{3+e^{2x}} \Longrightarrow$$

$$f''(x) = \frac{2e^{2x}(3+e^{2x}) - e^{2x}e^{2x}}{(3+e^{2x})^{3/2}} = \frac{6e^{2x}+e^{4x}}{(3+e^{2x})^{3/2}} > 0.$$

Then, $f(x)$ is convex on $(-\infty, 0)$.

c) The function is obviously increasing on $(-\infty, 0]$ and has a horizontal asymptote $y = \sqrt{3}$, so the graph of the function is approximately this:



(5) **Given the functions** $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, **defined by:** $f(x) = \dfrac{2x+2}{1-x}$, $g(x) = \sqrt{4-x}$, **then:**

(a) draw approximately the set A bounded by the graph of these functions and the x-axis $y = 0$. Using Pareto order, find if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

0.4 points part a); 0.6 points part b)

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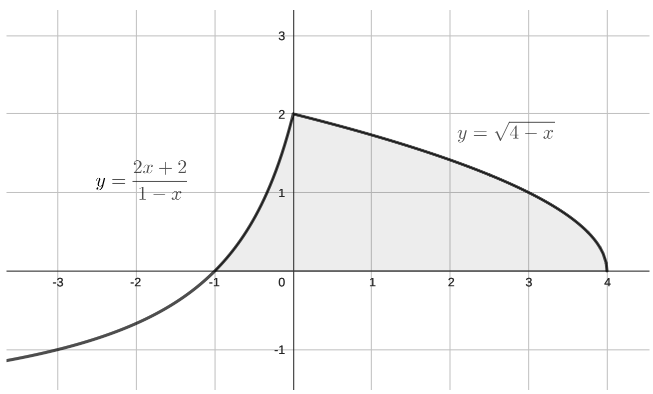
a) First of all, we can see that the line $y = 0$ intersect the graph of the functions: $f(x)$ at $x = -1$ and $g(x)$ at $x = 4$.

Secondly, $f(x)$ is increasing on $[-1, 1)$ (since $f(x) = \dfrac{2x-2+4}{1-x} = -2 + \dfrac{4}{1-x}$ then $f'(x) = \dfrac{4}{(1-x)^2} > 0$) and $g(x)$ is decreasing. So, their graphs only intersect in one point in the interval. In fact, at $x = 0$, since $f(0) = 2 = g(0)$.
Moreover, since $x > 1$, $f(x) < 0 \leq g(x)$, these graphs don't intersect each other in the interval $(1, 4]$.
So, the draw of A will be approximately like,



Then, Pareto order describes the set properties:
Minimum(A) = minimum elements$(A) = \{(-1, 0)\}$.
The maximum doesn't exist and maximal elements$(A) = \{(x, g(x)) : 0 \leq x \leq 4\}$.

b) First of all, looking at the position of the graphs we know that:
area(A)$=\int_{-1}^{0} f(x)dx + \int_{0}^{4} g(x)dx$.
Since $\int f(x)dx = \int(-2 + \frac{4}{1-x})dx = -2x - 4\ln(1-x)$, then applying Barrow's Rule we obtain:
$\int_{-1}^{0} f(x)dx = [-2x - 4\ln(1-x)]_{-1}^{0} = 0 - (-2(-1) - 4\ln(1-(-1)) = -2 + 4\ln 2$.
Moreover, since $\int g(x)dx = \int \sqrt{4-x}dx = -\frac{2}{3}(4-x)^{3/2}$, applying Barrow's Rule:
$\int_{0}^{4} g(x)dx = [-\frac{2}{3}(4-x)^{3/2}]_{0}^{4} = 0 - (-\frac{2}{3}(4-0)^{3/2}) = \frac{16}{3}$; then:
area(A)$=\int_{-1}^{0} f(x)dx + \int_{0}^{4} g(x)dx = -2 + 4\ln 2 + \frac{16}{3}$ area units.