Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	Total
Points						

Department of Economics **Mathematics I Final Exam** **January 12th 2022**

Exam time: 1 hour and 50 minutes.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = xe^{\frac{1}{x}}$**, defined in the interval** $(0, \infty)$**. Then:**

(a) find the asymptotes of the function and the intervals where $f(x)$ increases and decreases.

(b) find the global and local maximum and minimum, and range (or image) of $f(x)$. Draw the graph of the function.

(c) consider $f_1(x)$ to be the function $f(x)$ defined on the interval $[1, \infty)$. Sketch the graph of the inverse function of $f_1(x)$.

0.4 points part a); 0.4 points part b); 0.2 points part c)

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a) The domain of the function is $(0, \infty)$.
Since f is continuous on its domain, we only need to study its asymptotes at 0 on its right-hand side and at $+\infty$:
i) using the change of variable $x = \frac{1}{t}$ (in this case $x \longrightarrow 0^+$ if $t \longrightarrow +\infty$) we obtain:

$$\lim_{x \longrightarrow 0^+} f(x) = \lim_{t \longrightarrow \infty} f\left(\frac{1}{t}\right) = \lim_{t \longrightarrow \infty} \frac{e^t}{t} = \frac{\infty}{\infty} = [\text{using L' Hopital's Rule}] = \lim_{t \longrightarrow \infty} \frac{e^t}{1} = \infty$$

Therefore $f(x)$ has a right-sided vertical asymptote at $x = 0$.

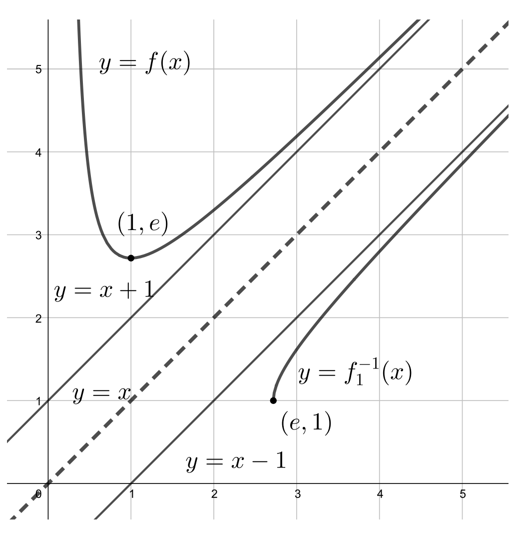
ii) $$\lim_{x \longrightarrow \infty} \frac{f(x)}{x} = \lim_{x \longrightarrow \infty} e^{\frac{1}{x}} = 1, \quad \text{and} \quad \lim_{x \longrightarrow \infty} f(x) - x = \lim_{x \longrightarrow \infty} x(e^{\frac{1}{x}} - 1) = \lim_{x \longrightarrow \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} =$$

$$= \frac{0}{0} = [\text{using L'Hopital's Rule}] = \lim_{x \longrightarrow \infty} \frac{e^{\frac{1}{x}}(-1/x^2)}{(-1/x^2)} = 1.$$

So, $f(x)$ has an oblique asymptote $y = x + 1$ at ∞.
Finally, since $f'(x) = e^{\frac{1}{x}}(1 - \frac{1}{x})$, we can deduce from the sign of $f'(x)$ that $f(x)$ is increasing on $[1, \infty)$, since $f'(x) > 0$ on the interval. Analogously, f is decreasing on $(0, 1]$ since $f'(x) < 0$.

b) Interpreting the monotonicity of f, it is deduced that $x = 1$ is a local and global minimizer. Furthermore, as there is not local maximizer then can not be a global one either.
On the other hand, since f is decreasing on $(0, 1]$, increasing on $[1, \infty)$ and $\lim_{x \longrightarrow 0^+} f(x) = \lim_{x \longrightarrow \infty} f(x) = \infty$, due to the Intermediate Value Theorem we can deduce that the range of the function in the interval $(0, \infty)$ will be $[f(1), \infty) = [e, \infty)$.
The graph of f will have an appearance approximately, similar to the one in the figure underneath.

c) We know that, f_1 is increasing on $[1, \infty)$, $f_1(1) = e$ and $f_1(x)$ has an oblique asymptote $y = x + 1$. Therefore, its inverse function is increasing on $[e, \infty)$, takes the value 1 at the point e, will have an oblique asymptote with equation $y = x - 1$ and the graph of its inverse will have an appearance approximately, similar to the one in this figure:



(2) **Given the implicit function $y = f(x)$, defined by the equation**

$$-3x + 3y + e^{-x} + e^{y} = 2$$

in a neighbourhood of the point $x = 0, y = 0$, it is asked:

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 0$.

(b) sketch the graph of the function f near the point $x = 0$.

(c) use the second-order Taylor Polynomial of $f(x)$ to obtain the approximate values of $f(-0.1)$ and $f(0,1)$. Will $f(0)$ be greater, less or equal than the exact value of $\frac{1}{2}(f(-0.1) + f(0.1))$?

(*Hint for part (b) and (c):* use $f''(0) < 0$).

0.4 points part a); 0.2 points part b); 0.4 points part c).

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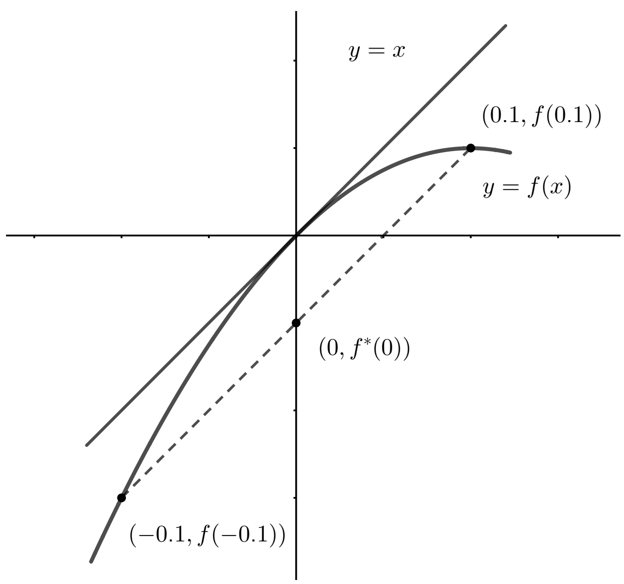
a) First of all, we calculate the first-order derivative of the equation:
$-3 + 3y' - e^{-x} + y'e^{y} = 0$
evaluating at $x = 0,\ y(0) = 0$ we obtain: $y'(0) = f'(0) = 1$.
Then the equation of the tangent line is: $y = P_1(x) = x$.
Secondly, we calculate the second-order derivative of the equation:
$3y'' + e^{-x} + y''e^{y} + (y')^2e^{y} = 0$
evaluating at $x = 0$, $y(0) = 0$, $y'(0) = 1$ we obtain: $y''(0) = f''(0) = -1/2$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = x - \frac{1}{4}x^2$.

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be as you can see in the figure underneath.

c) Finally, using the second-order Taylor Polynomial we obtain:
$f(-0.1) \approx -0.1 - \frac{1}{4}(-0.1)^2 = -0.1025$ and $f(0.1) \approx 0.1 - \frac{1}{4}(0.1)^2 = 0.0975 \Longrightarrow$
$\frac{1}{2}(f(-0.1) + f(0.1)) \approx -\frac{1}{4}(0.1)^2 = -0.0025$.
Finally, since $f(x)$ is concave, $\frac{1}{2}(f(-0.1) + f(0.1))$ will be less than $f(0)$, as you can notice looking at the graph below or if you prefer we can calculate its approximate value using the second-order Taylor Polynomial: $\frac{1}{2}(f(-0.1) + f(0.1)) \approx -0.0025$ is less than $f(0) = 0$
Naming $f^*(1) = \frac{1}{2}(f(0.9) + f(1.1))$, the graph will be:



(3) **Let** $C(x) = 36 + 16x + ax^2$ **be the cost function and** $p(x) = 76 - x$ **the inverse demand function of a monopolistic firm, with** $a > 0$**. Then:**

(a) Calculate the value of the parameter a, knowing that the production level to maximize the profit is $x^* = 6$.

(b) Calculate the value of the parameter a, knowing that the production level to minimize the average cost is $x^* = 6$.

0.5 points part a); 0.5 points part b).

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a) First of all, we calculate the profit function.
$B(x) = (76 - x)x - (36 + 16x + ax^2) = -(a+1)x^2 + 60x - 36$
Secondly, we calculate the first and second order derivatives of B:
$B'(x) = -2(a+1)x + 60$; $B''(x) = -2(a+1) < 0$
we see that B has a unique critical point at $x^* = \dfrac{60}{2(a+1)}$
and, since B is a concave function, the critical point is the unique global minimizer.
Finally, $x^* = 6 = \dfrac{60}{2(a+1)} \Longrightarrow a + 1 = 5 \Longrightarrow a = 4$

b) The average cost function is $\dfrac{C(x)}{x} = \dfrac{36}{x} + 16 + ax$,
its first order derivative: $\left(\dfrac{C(x)}{x}\right)' = -\dfrac{36}{x^2} + a = 0 \Longleftrightarrow x^2 = \dfrac{36}{a}$.
Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{72}{x^3} > 0$, the function is convex and the critical point will be the global minimizer.
Then $x^* = 6 = \dfrac{6}{\sqrt{a}} \Longrightarrow a = 1$

(4) **Let**

$$f(x) = \begin{cases} x^2 - 2x + a & , x \leq 1 \\ bx^2 + 2x + 1 & , x > 1 \end{cases}$$

be a piece-wise defined function in the interval $[0, 2]$. **Then:**

(a) state Bolzano's Theorem for any function f defined in the interval $[0, 2]$. Calculate a and b such that $f(x)$ satisfies the hypotheses (or conditions) of this theorem.
Are the hypotheses (or initial conditions) satisfied for any $b < 0$?

(b) state Lagrange's Theorem (or Mean Value Theorem) for any function f defined in the interval $[0, 2]$. Find a, b that satisfy the hypotheses of the theorem.
For the found values of a, b calculate the values of c that satisfy the thesis or conclusion of the theorem.
0.5 points part a); 0.5 points part b)

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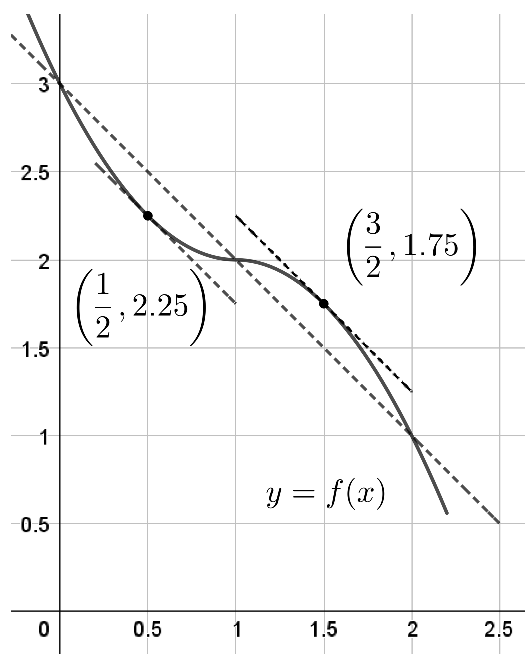
a) The hypotheses are f is continuous on $[0, 2]$ and $f(0) \cdot f(2) < 0$.
The thesis or conclusion is, there is a $c \in (0, 2)$ such that $f(c) = 0$.
First of all, we need that the function f is continuous at $x = 1$. Since, $\lim_{x \longrightarrow 1^-} f(x) = f(1) = -1 + a$ and $\lim_{x \longrightarrow 1^+} f(x) = b+3$, we can deduced that the function will be continuous in $[0, 2]$ when: $a = b+4$ or if you prefer, $b = a - 4$.
Secondly, supposing f continuous ($b = a - 4$), the condition $f(0) \cdot f(2) < 0$ will be satisfied, when:
i) $f(0) = a < 0$ and $f(2) = 4b + 5 > 0$; or
ii) $f(0) = a > 0$ and $f(2) = 4b + 5 < 0$.
So, in the first case if $a < 0$ (or $b < -4$ since $b = a - 4$), we need $f(2) = 4b + 5 > 0 \Longrightarrow b > \frac{-5}{4}$ which is impossible. Then in the first case i), when $a < 0$, the hypothesis is never satisfied.
And in the other case, if $a > 0$ (or $b > -4$ since $b = a - 4$), we need $f(2) = 4b + 5 < 0 \Longrightarrow b < \frac{-5}{4}$ and we obtain the solution $-4 < b < \frac{-5}{4} < 0$ or equivalently $0 < a < \frac{11}{4}$, that satisfies the initial conditions for the theorem in case ii).

b) The hypotheses are that f is continuous on $[0, 2]$ and differentiable on $(0, 2)$.
The thesis (or conclusion) is that there is a $c \in (0, 2)$ such that $f(2) - f(0) = 2f'(c)$.
We have seen that the function is continuous on $[0, 2]$ when: $a = b + 4$.
Supposing the continuity of f, we need it to be differentiable at $x = 1$.

Since $f'(x) = \begin{cases} 2x - 2 & , x < 1 \\ 2bx + 2 & , x > 1 \end{cases}$ then $\lim_{x \longrightarrow 1^-} f'(x) = 0$, $\lim_{x \longrightarrow 1^+} f'(x) = 2b + 2$
Supposing the continuity of $f'(x)$ at $x = 1$ when $2b+2 = 0$. The hypotheses of the theorem are satisfied if $b = -1$, $a = 3$. Now, with those values the point c must satisfy $f(2) - f(0) = -2 = 2f'(c)$. Obviously, $c \neq 1$; so then:
i) if $c < 1 : 2c - 2 = -1 \Longrightarrow c = \frac{1}{2}$
ii) if $c > 1 : -2c + 2 = -1 \Longrightarrow c = \frac{3}{2}$
And the situation can be seen at the figure.



(5) **Given the functions** $f, g : [1,2] \longrightarrow \mathbb{R}$, **defined by:** $g(x) = -e^{x-1}$, $f(x) = x \ln x$, **then:**

(a) draw approximately the set A, bounded by the graph of these functions and the straight lines $x = 1, x = 2$.

Find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

Hint for part (a): Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1,\ y_0 \leq y_1$.

0.4 points part a); 0.6 points part b)

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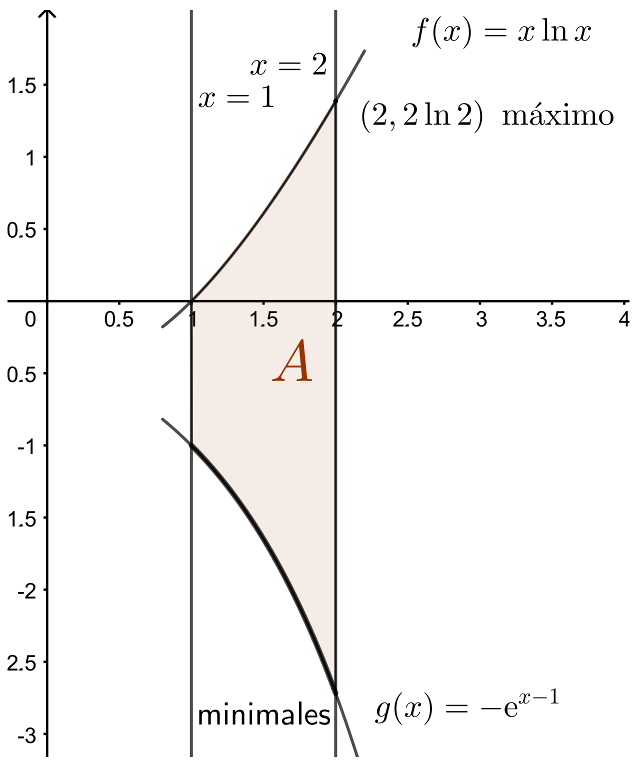
a) $f(x)$ is increasing on $[1,2]$ (since $f'(x) = 1 + \ln x > 0$) and $g(x)$ is decreasing (since $g'(x) = g(x) < 0$).

Moreover, since $g(1) = -1 < 0 = f(1)$, then:

$g(x) < f(x)$ if $1 < x < 2$

So, the draw of A will be approximately like,



Then, Pareto order describes the set properties:

maximum(A) = maximal elements$(A) = (2, 2\ln 2) = (2, \ln 4)$.

The minimum doesn't exist and minimal elements$(A) = \{(x, g(x)) : 1 \leq x \leq 2\}$.

b) First of all, looking at the position of the graphs we know that:

area(A)$=\int_1^2 (f(x) - g(x))dx = \int_1^2 (x \ln x - (-e^{x-1}))dx$.

We calculate a primitive of $f(x)$ integrating by parts:

$$\int x \ln x dx = (\text{naming } v' = x,\ u = \ln x) = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

and, since the other integral is basic, then applying Barrow's Rule we obtain:

$$\int_1^2 (x \ln x + e^{x-1})dx = \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} + e^{x-1} \right]_1^2 = 2\ln 2 - 1 + e - (-\frac{1}{4} + 1) =$$

$$= \ln 4 + e - \frac{7}{4} \text{ area units.}$$