Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	6	Total
Points							

Department of Economics **Mathematics I Final Exam** **January 26th 2021**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = (x+1)^2 e^{-x}$. Then:**

(a) find the asymptotes of the function and the intervals where $f(x)$ increases and decreases.

(b) find the global maximum and minimum, and range (or image) of $f(x)$. Draw the graph of the function.

(c) consider $f_1(x)$ to be the function $f(x)$ defined on the interval $[-1,1]$, sketch the graph of the inverse function of $f_1(x)$.

(*Hint for part (c):* do not try to calculate the explicit formula of the inverse function of f_1)

0.4 points part a); 0.4 points part b); 0.2 points part c)

(a) The domain of the function is $\mathbb{R}$.

Since f is continuous on its domain, we only need to study its asymptotes at ∞ and $-\infty$:

i) $\lim\limits_{x\longrightarrow\infty} f(x) = \lim\limits_{x\longrightarrow\infty} \dfrac{(x+1)^2}{e^x} = \frac{\infty}{\infty} =$ [applying L'Hopital's Rule twice] $= \lim\limits_{x\longrightarrow\infty} \dfrac{2}{e^x} =$
$= \dfrac{2}{\infty} = 0$. Therefore $f(x)$ has a horizontal asymptote $y = 0$ at ∞.

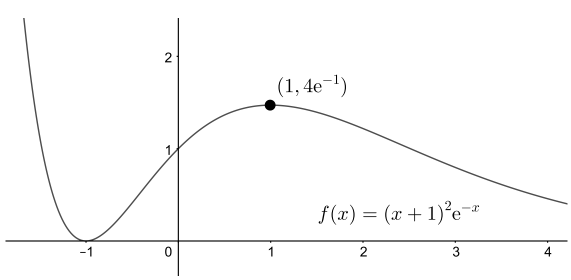
ii) $\lim\limits_{x\longrightarrow-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\longrightarrow-\infty} \dfrac{(x+1)^2}{x}. \lim\limits_{x\longrightarrow-\infty} e^{-x} = -\infty$, then f has no horizontal neither oblique asymptote at $-\infty$.

As $f'(x) = e^{-x}(1-x^2)$, we can deduce: f is increasing $\Longleftrightarrow f'(x) > 0 \Longleftrightarrow 1 - x^2 > 0$; then f is increasing on $[-1,1]$. Analogously, f is decreasing on $(-\infty,-1]$ and $[1,\infty)$.

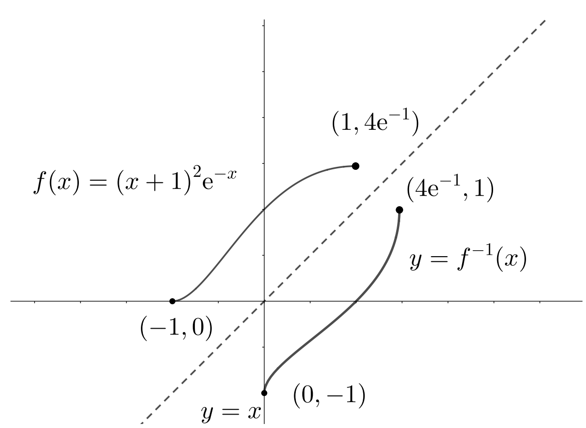
(b) Interpreting the monotonicity of f, it is deduced that -1 is a local minimizer and 1 is a local maximizer. Since $\lim\limits_{x\longrightarrow-\infty} f(x) = \infty$, there is no global maximum. In addition, as $f(-1) = 0$ and $f(x) > 0$ (if $x \neq -1$), it is deduced that 0 is a strict (unique) global minimizer. Finally, as $f(-1) = 0, f(x) \geqslant 0$ and $\lim\limits_{x\longrightarrow-\infty} f(x) = \infty$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $[0,\infty)$.

The graph of f will have an appearance approximately, similar to the one in figure A.

(c) We know that, f_1 is increasing on $[-1,1]$, $f_1(-1) = 0$, $f_1(1) = 4/e$. Therefore, the graph of its inverse will have an appearance approximately, similar to the one in figure B:



(A)



(B)

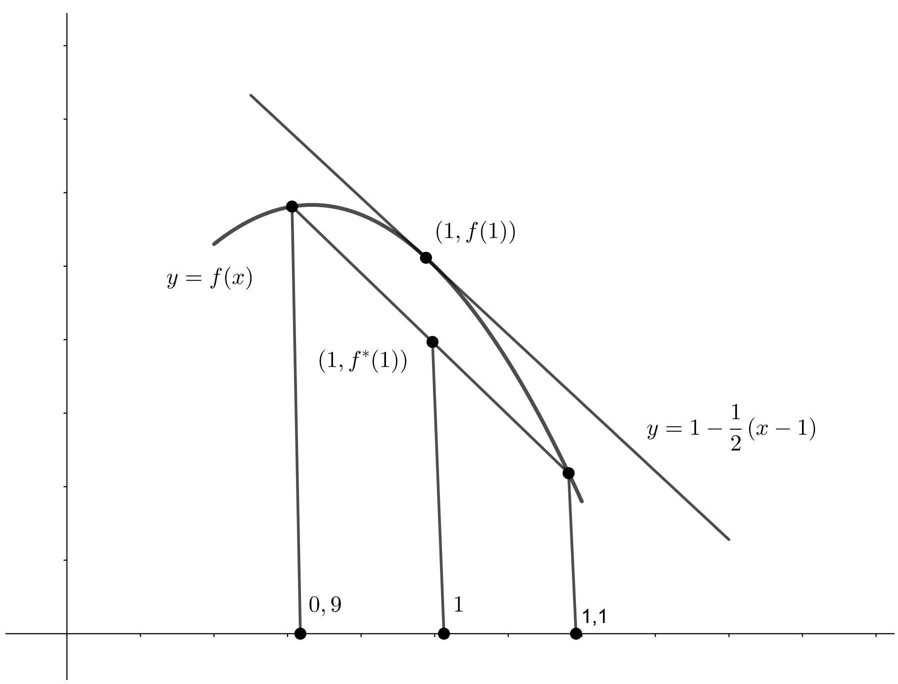
(2) **Given the implicit function $y = f(x)$, defined by the equation $e^x + ye^y = 2e$ in a neighbourhood of the point $x = 1, y = 1$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 1$.

(b) sketch the graph of the function f near the point $x = 1, y = 1$. Use the tangent line to the graph of $f(x)$ to obtain the approximate values of $f(0.9)$ and $f(1.1)$.
Will $f(1)$ be greater, less or equal than the exact value of $\frac{1}{2}(f(0.9) + f(1.1))$?
(*Hint for part (b):* use that $f''(1) < 0$.
0.5 points part a); 0,5 points part b)

(a) First of all, we calculate the first-order derivative of the equation:
$e^x + y'e^y + yy'e^y = e^x + y'(y + 1)e^y = 0$
evaluating at $x = 1$, $y(1) = 1$ we obtain: $y'(1) = f'(1) = -1/2$.
Then the equation of the tangent line is: $y = P_1(x) = 1 - \frac{1}{2}(x - 1)$. Secondly, we calculate the second-order derivative of the equation:
$e^x + y''(y + 1)e^y + (y')^2e^y + y'(y + 1)y'e^y = 0$
evaluating at $x = 1$, $y(1) = 1$, $y'(1) = -1/2$ we obtain $y''(1) = f''(1) = -7/8$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 1 - \frac{1}{2}(x - 1) - \frac{7}{16}(x - 1)^2$.

(b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 1$, will be as you can see in the figure underneath. On the other hand, using the tangent line, the first order approximation will be:
$f(1.1) \approx 1 - \frac{1}{2}(0.1) = 0.95$; $f(0.9) \approx 1 - \frac{1}{2}(-0.1) = 1.05$.
Finally, since $f(x)$ is concave, $\frac{1}{2}(f(0.9) + f(1.1))$ will be less than $f(1)$, as you can notice looking at the graph below or if you prefer we can calculate its approximate value using the second-order Taylor Polynomial: $\frac{1}{2}(f(0.9) + f(1.1)) \approx 1 - \frac{7}{8}0.01 < f(1) = 1$.
Naming $f^*(1) = \frac{1}{2}(f(0.9) + f(1.1))$, the graph will be:



(3) **Let** $C(x) = C_0 + 50x + \frac{1}{2}x^2$ **be the cost function and** $p(x) = 710 - 5x$ **the inverse demand function of a monopolistic firm. Then:**

(a) calculate the price p^* and the production x^* that maximizes the profit.

(b) find C_0 such that the production obtained in part a) would be the same that minimizes the average cost.

0.5 points part a); 0.5 points part b)

(a) First of all, we calculate the profit function.

$B(x) = (710 - 5x)x - (C_0 + 50x + \frac{1}{2}x^2) = -\frac{11}{2}x^2 + 660x - C_0$

Secondly, we calculate the first and second order derivatives of B:

$B'(x) = -11x + 660;\ B''(x) = -11 < 0$

we see that B has a unique critical point at $x^* = \dfrac{660}{11} = 60$ and, since B is a concave function, the critical point is the unique global minimizer.

Finally, $p^* = p(60) = 710 - 300 = 410$

(b) The average cost function is $\dfrac{C(x)}{x} = \dfrac{C_0}{x} + 50 + \frac{1}{2}x$,

its first order derivative: $\left(\dfrac{C(x)}{x}\right)' = -\dfrac{C_0}{x^2} + \frac{1}{2} = 0 \Longleftrightarrow x^2 = 2C_0$.

Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{2C_0}{x^3} > 0$, the function is convex and the critical point will be the global minimizer.

Since $x^* = 60$ must be the minimizer, the solution will be

$60 = x^* = \sqrt{2C_0} \Longrightarrow C_0 = 1800.$

(4) **Let** $f(x) = \begin{cases} (x+a)^2, & x < 2 \\ b, & x = 2 \\ -x^2+6x+1, & x > 2 \end{cases}$ be a piece-wise defined function in the interval $[1,3]$. **Then:**

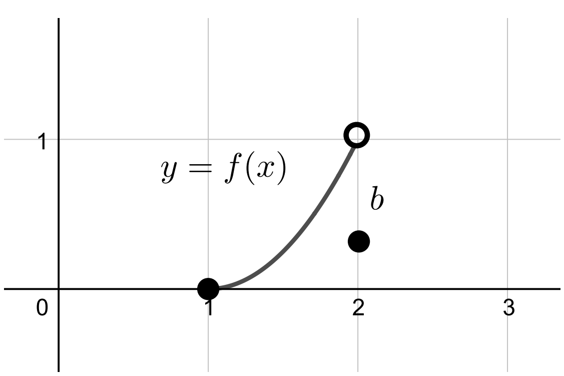
(a) state Weierstrass' Theorem for a function g defined in an interval I. Calculate a y b such that $f(x)$ satisfies the hypothesis of this theorem.

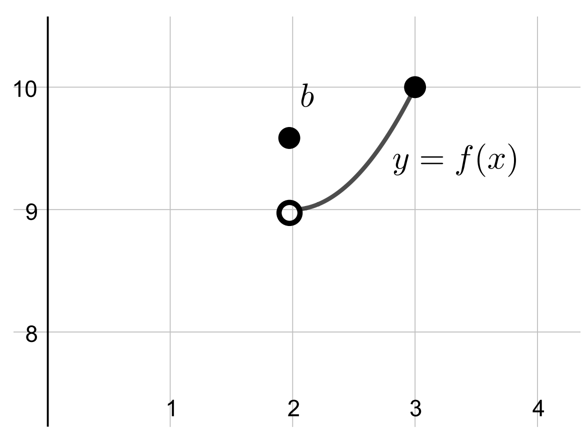
(b) suppose that $a = -1$, find the values of b such that the thesis (or conclusion) of Weierstrass' Theorem is satisfied in the interval $[1,3]$. What can you say for the intervals $[1,2]$ or $[2,3]$?

0.4 points part a); 0.6 points part b)

(a) The hypothesis is that g is continuous in an interval I closed and bounded. The thesis (or conclusion) is that the function g attains its global maximum and minimum on I.
Thus, we need that the function f is continuous at $x = 2$.
Since, $\lim_{x \longrightarrow 2^+} f(x) = -4 + 12 + 1 = b = f(2) \Longrightarrow b = 9$.
And $\lim_{x \longrightarrow 2^-} f(x) = (2+a)^2 = 9 = f(2) \Longrightarrow a = -5$ or $a = 1$.
Therefore, we can deduced that the function will be continuous in $[1,3]$ when: $b = 9$ and ($a = -5$ or $a = 1$).

(b) For the value $a = -1$ the hypothesis of the theorem is not satisfied in the interval $[1,3]$.
Meanwhile, it could be possible that the thesis is satisfied in this interval depending on the values of b.
If we notice that f is increasing in $[1,2)$ and also in $(2,3]$, and furthermore:
$0 = f(1) < \lim_{x \longrightarrow 2^-} f(x) = 1 < 9 = \lim_{x \longrightarrow 2^+} f(x) < f(3) = 10$.
We can consider three different cases depending on b:
i) $b \leq 0 \Longrightarrow \min f = b, \max f = 10$.
ii) $0 \leq b \leq 10 \Longrightarrow \min f = 0, \max f = 10$.
iii) $10 \leq b \Longrightarrow \min f = 0, \max f = b$.
Then, for any real value of b the thesis of Weierstrass' Theorem is satisfied.
Now, in the case of the interval $[1,2]$ the theorem is only satisfied if $b \geqslant 1$, and it happens that $\min f = 0, \max f = b$. Notice that if $b < 1$ the maximum doesn't exist as we can appreciate in the left graph below.
Analogously, in the case of the interval $[2,3]$ the theorem is only satisfied if $b \leq 9$, and it happens that $\min f = b$, $\max f = 10$. Notice that if $b > 9$ the minimum doesn't exist, as we can appreciate in the right graph below.





(5) **Given the functions** $f, g : [-1, 8] \longrightarrow \mathbb{R}$, **defined by:** $f(x) = -\frac{1}{2}x + 1$, $g(x) = \sqrt{x+1}$, **then:**

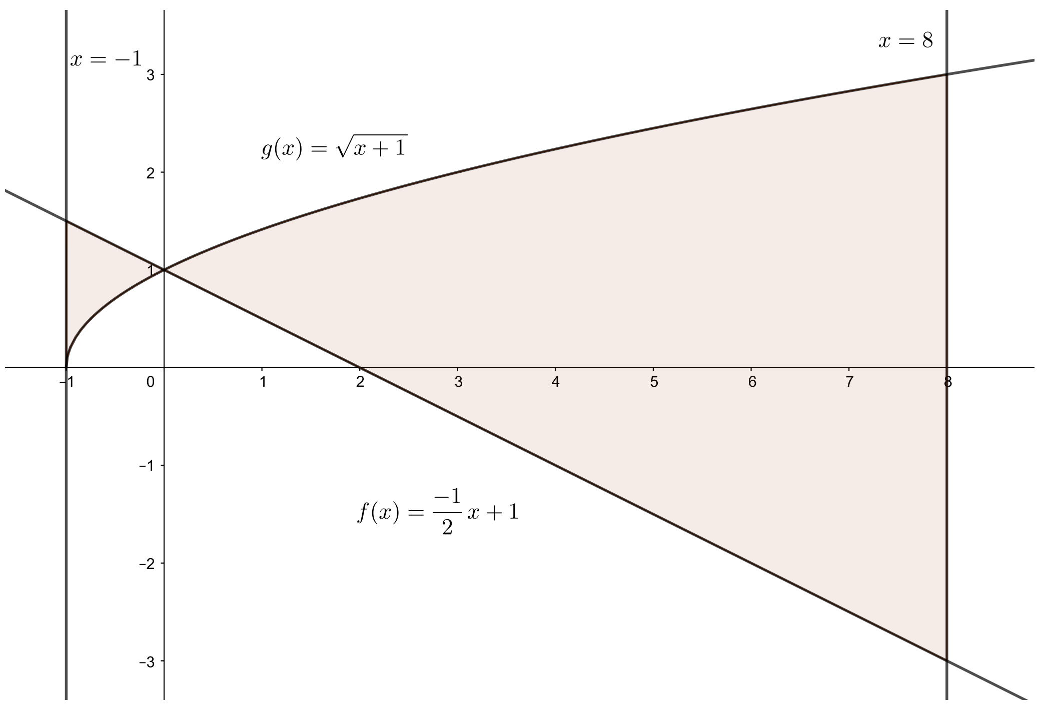
(a) draw approximately the set A, bounded by the graph of these functions and the straight lines $x = -1$, $x = 8$. Find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set (expressing the result as a whole number plus a fraction between 0 and 1).

Hint for part (a): Pareto order is defined as: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1, y_0 \leq y_1$.

0.6 points part a); 0.4 points part b)

(a) $f(x)$ and $g(x)$ are increasing functions and $f(0) = g(0) = 1$ then:
i) $g(x) < f(x)$ if $-1 < x < 0$; and
ii) $f(x) < g(x)$ if $0 < x < 8$.
So, the draw of A will be approximately like,



Then, Pareto order describes the set properties: maximum(A) = maximal elemements$(A) = (8, 3)$.
Also, since $g(-1) = 0 = f(2)$, then:
the minimum doesn't exist and minimal elements$(A) = \{(-1, 0)\} \cup \{(x, f(x)) : 2 < x \leq 8\}$.

(b) First of all, looking at the position of the graphs we know that:

area(A)$= \int_{-1}^{0} (f(x) - g(x))dx + \int_{0}^{8} (g(x) - f(x))dx = \int_{-1}^{0} (-\frac{1}{2}x + 1 - \sqrt{x+1})dx + \int_{0}^{8} (\sqrt{x+1} + \frac{1}{2}x - 1)dx =$

Then applying Barrow's Rule we obtain:

$= [-\frac{1}{4}x^2 + x - \frac{2}{3}(x+1)^{3/2}]_{-1}^{0} + [\frac{2}{3}(x+1)^{3/2} + \frac{1}{4}x^2 - x]_0^8 =$

$= [-\frac{2}{3} - (-\frac{1}{4} - 1)] + [\frac{2}{3}9^{3/2} + \frac{1}{4}8^2 - 8 - \frac{2}{3}] = 25 + \frac{11}{12}$ area units.

(6) **Given the function** $f(x) = \dfrac{3x+5}{x+1}$. **Then:**

(a) calculate $\int_0^1 f(t)dt$.

(b) suppose now that we interchange $f(x)$ by a decreasing convex function $g(x)$ so that $g(0) = 5$, $g(1) = 4$. Whith the provided information, find the greatest A_1 and the least B_1 such that $A_1 < \int_0^1 g(x)dx < B_1$.

(c) finally, suppose that $g(x)$ is a convex function such that $g(0) = 5$, $g(1) = 4$, $g'(1) = -\frac{1}{2}$. Find $A_2 > A_1$ so that $A_2 < \int_0^1 g(x)dx$. Discuss if the function $g(x)$ will be decreasing or not.

Hint for part (b): draw $f(x)$ and the segment that joint the points $(0, f(0))$ with $(1, f(1))$.

Hint for part (c): draw $f(x)$ and its tangent line at $x = 1$.

0.2 points part a); 0.4 points part b); 0.4 points part c).

(a) $\int_0^1 f(x)dx = \int_0^1 (3 + \frac{2}{x+1})dx = [3x + 2\ln(x+1)]_0^1 = 3 + 2\ln 2$.

(b) i) First of all, since $g(x)$ is decreasing, $g(1) = 4 < g(x) \Longrightarrow 4 < \int_0^1 g(x)dx$. Then $A_1 = 4$

ii) Secondly, $g(x)$ is underneath the line segment joining the points $(0, 5)$ y $(1, 4)$.

The equation of this line is $y = 5 - x$, then $\int_0^1 g(x)dx < 5 - \frac{1}{2} = 4 + \frac{1}{2}$.

The draw of $g(x)$, together with the line segment can be seen in the left graph below. Then, $B_1 = 4 + \frac{1}{2}$

(c) i) First of all, $g(x)$ is decreasing in $[0, 1]$, since $g'(x)$ is increasing (g is a convex function), then $x < 1 \Longrightarrow g'(x) < g'(1) < 0$.

ii) Secondly, $g(x)$ is above the line $y = g(1) + g'(1)(x-1) = 4 - \frac{1}{2}(x-1)$, tangent to $g(x)$ at $x = 1$.

Then, $\int_0^1 (4 - \frac{1}{2}(x-1))dx < \int_0^1 g(x)dx \Longrightarrow 4 + \frac{1}{4} < \int_0^1 g(x)dx$.

The draw of $g(x)$, together with the tangent line at $x = 1$ can be seen in the right graph below.

Being the solution $A_2 = 4 + \frac{1}{4}$