WORKSHEET 4 : Differentiation II

1. Compute the following limits:

   a)(*) $\lim_{x\to\infty}(1+x)^{1/x}$ b) $\lim_{x\to 0^+} x\mathrm{ln}x$ c)(*) $\lim_{x\to\infty} x^{1/x}$

   d)(*) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{2}{x-1}\right)$ e) $\lim_{x\to\infty} x\tan(1/x)$ f) $\lim_{x\to 0}\frac{arcsinx-\arctan x}{x}$

   g) $\lim_{x\to 1/2}(4x^2-1)\tan(\pi x)$

a) $\lim_{x\to\infty}(1+x)^{1/x}=1$. Analogously, part c).
b) $\lim_{x\longrightarrow 0^+} x\ln x=0$
d) $\lim_{x\to 1^+}\frac{1}{\ln x}-\frac{2}{x-1}=-\infty$

2. Compute the asymptotes of the following functions:

   a)(*) $f(x)=\frac{2x^3-3x^2-8x+4}{x^2-4}$ b) $f(x)=\frac{x^3}{x^3+x^2+x+1}$ c)(*) $f(x)=2x+e^{-x}$

   d) $f(x)=\frac{sinx}{x}$ e)(*) $f(x)=\frac{x-2}{\sqrt{4x^2+1}}$ f) $f(x)=\frac{3x^2-x+2\sin x}{x-7}$

   g)(*) $f(x)=\frac{e^x}{x}$ h)(*) $f(x)=xe^{1/x}$ i)(*) $f(x)=\frac{x}{e^x-1}$

a) Vertical asymptotes in $x=2$ and in $x=-2$.
On the other hand, the oblique asymptote in ∞ and in $-\infty$ is $y=2x-3$.
c) $y=2x$ is the oblique asymptote in ∞.
e) $\lim_{x\to\infty}\frac{x-2}{\sqrt{4x^2+1}}=\frac{1}{2}$, $\lim_{x\to-\infty}\frac{x-2}{\sqrt{4x^2+1}}=-\frac{1}{2}$. There are no more asymptotes.
g) $\lim_{x\to 0^+}\frac{e^x}{x}=\infty$, $\lim_{x\to 0^-}\frac{e^x}{x}=-\infty$, and there are no more vertical asymptotes.
On the other hand, $y=0$ is horizontal asymptote in $-\infty$, and there are no horizontal, nor oblique asymptote in ∞.
h) $\lim_{x\to 0^+} xe^{1/x}=\infty$, and there are no more vertical asymptotes.
On the other hand, $y=x+1$ is the oblique asymptote in ∞, and also in $-\infty$.
i) There is no vertical asymptote.
On the other hand, $y=0$ is the horizontal asymptote in ∞. Finally, the line $y=-x$ is the oblique asymptote in $-\infty$.

3. (*)Find the Taylor polynomium of order 2 in a and, using that polynomium, compute the approximate value of the function on $x=a+0.1$.

   a) $f(x)=e^x$ in $a=0$ b) $f(x)=\sin x$ in $a=0$ c) $f(x)=\frac{\ln x}{x}$ in $a=1$

a) $P(x)=1+x+x^2/2$, so $f(0.1)\approx 1.105$
b) $P(x)=x$, so $f(0.1)\approx 0.1$
c) $P(x)=(x-1)-3\frac{(x-1)^2}{2}$, so $f(1.1)\approx 0.085$

4. (*)Given the Taylor polynomium of order 2 in $a=0$ of f , determine if the function has a local maximum or minimum at the point $(0,f(0))$.

   a) $P(x)=1+2x^2$ b) $P(x)=1+x+x^2$ c) $P(x)=1-2x^2$

a) f has a local minimum at the point $(0,f(0))$.
b) f has not a local maximum or mínimum at the point $(0,f(0))$.
c) f has a local maximum at the point $(0,f(0))$.

5. Compute the (absolute and local) maxima and minima of f in the given intervals:

   a)(*) $f(x)=3x^{2/3}-2x$ in $[-1,2]$.

   b) $f(x)=xe^{-x}$ in $[1/2,\infty)$, $[0,\infty)$ and $\mathbb{R}$.

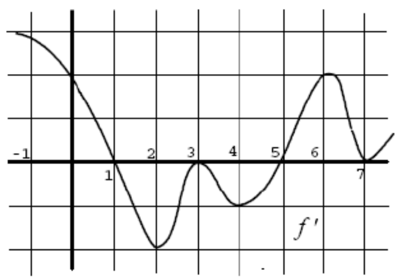
   a)

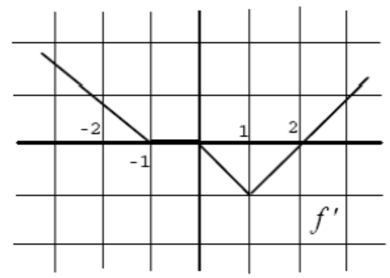
i) f obtains a local minimum in x=0 and a local maximum in x=1.
ii) f obtains its absolute minimum in $x = 0$.
iii) f obtains its absolute maximum in $x = -1$.

6. (*)Compute in which point the slope of the tangent line to the graph of the function $f(x) = -x^3 + 2x^2 + x + 2$ takes its maximum value.

   $x = \frac{2}{3}$

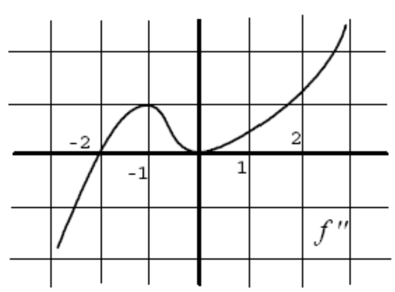
7. The first (*) and second drawings show the graphs of the derivatives of different functions f. Determine the increasing/decreasing, concavity/convexity intervals of f , and its local extreme and inflection points.





a) f is increasing in $(-\infty, 1]$ and in $[5, \infty)$.
   f is decreasing in $[1, 5]$.
   So, f obtains a local maximum in 1 and a local minimum in 5. On the other hand,
   f is convex in [2,3], [4,6] and in [7,∞);
   f is concave in ($-\infty$,2], [3,4] and in [6,7].
   So f has inflection points in 2, 3, 4, 6 y 7.
b) f is increasing in $(-\infty, -1]$ and in $[2, \infty)$.
   f is decreasing in $[0, 2]$.
   Finally, f is constant in $[-1, 0]$.
   Therefore, f reaches a relative maximum at all the points of the interval $[-1, 0]$.
   Analogously f reaches a relative minimum at all the points of the interval $(-1, 0)$ and at the point 2. On the other hand,
   f is convex in [1,∞);
   f is concave in ($-\infty$,-1] and in [0,1].
   Therefore, f has an inflection point at 1.

8. The following drawing shows the graph of the second derivative of f. Determine the convexity intervals of f and the inflection points. Determine where the function is increasing and decreasing and the relative extrema of f assuming that $f\prime(-3) = f\prime(0) = 0$.



f is convex in [-2, ∞). f is concave in $(-\infty, -2]$.
Therefore, f has an inflection point in x=-2.
Furthermore, f is increasing in $(-\infty, -3]$. Analogously, f is decreasing in $[-3, 0]$.
In the same way, f is increasing in $[0, \infty)$.
Therefore, f has a local maximum at x=-3 and a local minimum at x=0.

9. Let $f(x) = \begin{cases} x^\alpha \text{ if } 0 \le x \le 1 \\ x^\beta \text{ if } 1 \le x \end{cases}$ Discuss, depending on the values of α y β,when f is concave or convex.

   f will be convex in $[0, \infty)$ when $1 < \alpha \le \beta$.

   Analogously, f will be concave in $[0, \infty)$ when $0 < \beta \le \alpha < 1$.

10. (*)Let $f : \mathbb{R} \to \mathbb{R}$ convex, and let $x > 0$. Check graphically the following inequalities:

$$f(1) < \frac{1}{2}\left(f(1-x) + f(1+x)\right) < \frac{1}{2}\left(f(1-2x) + f(1+2x)\right)$$

11. (*)Let $f : \mathbb{R} \to \mathbb{R}$ concave, and let $x > 0$. Check graphically the following inequalities:

$$f(1) > \frac{1}{2}\left(f(1-x) + f(1+x)\right) > \frac{1}{2}\left(f(1-2x) + f(1+2x)\right)$$

12. (*)Let $f : [0, \infty) \to \mathbb{R}$, convex, such that $f'(1) = 0$.

a) Find the local extrema of f.
b) What can be said of the global extrema of f?
c) Consider now $f : [0, n] \to \mathbb{R}$. What can be said of the global extrema of f ?
a) and b): 1 is a local and global minimum of f
Moreover, it cannot exist a global maximizer of f, since $\lim_{x\to\infty} f(x) = \infty$.

c) In that case, in addition to what we said about minimizers, we can guarantee that it will exist a global maximizer, that will be the point 0 (if $f(n) \leq f(0)$) or the point n (if $f(0) \leq f(n)$).

13. (*)Let $f : [0, \infty) \to \mathbb{R}$, concave, such that $f'(1) = 0$.

a) Find the local extrema of f.
b) What can be said of the global extrema of f?
c) Consider now $f : [0, n] \to \mathbb{R}$. What can be said of the global extrema of f?
The same as the previous problem, changing maximum for minimun and viceversa.

14. Study and graph the following functions:

a) $f(x) = x + \cos x$ b) $f(x) = \dfrac{e^{2x}}{e^x - 1}$ c) $f(x) = \dfrac{x}{\ln x}$ d) $f(x) = \sqrt{|x-4|}$

Solution:

a) $f(x)$ is always increasing and the only root of this function is on the interval $(-\frac{\pi}{2}, 0)$.

Finallly:

$f(x)$ is convex in the intervals like $(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi)$, where $k = ..., -1, 0, 1, ...$

$f(x)$ is concave in the intervals like $(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi)$, where $k = ..., -1, 0, 1, ...$

And, consequently, the inflection points of $f(x)$ are like $\frac{\pi}{2} + k\pi$, where $k = ..., -1, 0, 1, ...$

15. (*)Given the cost function $C(x) = 4000 + 10x + 0.02x^2$ and the demand function $p(x) = 100 - (x/100)$, find the price p per unit that gives the maximum benefit.

$p = 85$.

16. (*)Let $p(x) = x^2 - x + 1/3$ be the sale price of 1 kilo of plutonium when x units are sold. Taking into account that the firm sells in the market a maximum of 2 kilos, find the value of x that maximizes the profits of the firm. We can assume that the Government pays all the costs of the firm.

The maximum is reached at the point x=2.

17. (*)Let $p(x) = 100 - x^2/2$ be the demand function of a product and $C(x) = 48 + 4x + 3x^2$ its cost function. What is the production x that minimizes the average cost? And if there exists a maximum production $x\hat{}$?

$x = 4$ on the first case, $x = \min(4, x\hat{})$ on the second case.

18. A firm that has a cost function $c(x) = x^2+1$ faces a demand given by the function $p(x) = \begin{cases} 10 & \text{si } 0 \leq x \leq 1 \\ 1 & \text{si } 1 < x \leq 10 \end{cases}$.

Find the production that gives the maximum profit.

19. (*)A manufacturer sells 5000 units per month for 100 euros per unit and he believes that his sales would increase by 500 units for each 5 euros of decrease on the unitary price.

a) Find the demand, revenues and marginal revenues functions.

b) If the cost of production of x units is $C(x) = 1000 + 0.12x$, find the marginal profit function.

The demand function is $p(x) = 150 - 0'01x$.

$I(x) = 150x - 0'01x^2$, $I\,'(x) = 150 - 0'02x$.

b) $B'(x) = 149'88 - 0'02x$.