WORKSHEET 1: Introduction

1. For each one of the following inequalities, determine the set of real numbers that satisfy them. Draw that set.

   a)(*) $|9-2x|<1$ b)(*)$-5|x+3|<4x-5$ c)(*)$\frac{|x|}{3}+2<|x|$ d)(*) $1<|3-2x|$

   e)(*) $\frac{(x^2-16)(x-1)}{x-3}\geq 0$ f) $|x-3|+|x+3|<10$ g) $|x-3|+|x+3|<\alpha,\ \alpha\in\mathbb{R}$ h)$|\frac{x-1}{x}|-1\geq 0$

a) $Sol:(4,5)$
b) $Sol:(-\infty,-20)\cup(-10/9,\infty)$.
c) $Sol:(-\infty,-3)\bigcup(3,\infty)$.
d) $Sol:(-\infty,1)\cup(2,\infty)$.
e) $Sol:(-\infty,-4]\cup[1,3)\cup[4,\infty)$.
f) $Sol.:(-5,5)$.
g) If $\alpha\leq 6$, the solution is the empty set; if $\alpha>6$, the solution is $(-\alpha/2,\alpha/2)$.
h) Sol.: $(-\infty,0)\cup(0,\frac{1}{2}]$.

2. (*)Interpret geometrically the inequalities a), b), c) and d) using the functions

   a) $y=|9-2x|$; $y=1$ b) $y=-5|x+3|$; $y=4x-5$
   c) $y=\frac{|x|}{3}+2$; $y=|x|$ d) $y=1$; $y=|3-2x|$

3. Discuss if the following inequalities are satisfied :

   a) (*) $|x+y|\leq|x|+|y|$ b)(*) $|x|+|y|\leq|x+y|$ g)(*) $|x-y|\leq|x|+|y|$
   c)(*) $|x-y|\leq|x|-|y|$ d)(*) $|x|-|y|\leq|x-y|$ h)(*) $|x|+|y|\leq|x-y|$
   e)(*) $||x|-|y||\leq|x|+|y|$ f) $|x|+|y|\leq||x|-|y||$ i) $||x|-|y||\leq|x|-|y|$

a) It is always true.
b) It is only satisfied if $x,y\in[0,\infty)$ or if $x,y\in(-\infty,0]$.
c) It is only satisfied if $0\leq y\leq x$ or $x\leq y\leq 0$.
d) It is always satisfied.
e) It is always satisfied.
f) It is only satisfied if $x=0$or $y=0$.
g) It is always satisfied.
h) It is only satisfied when x and y have different sign.
i) It is only satisfied if $|y|\leq|x|$.

4. Discuss if the following statements are true or false

   a) $x<y\Rightarrow x^2<\ y^2$ b) $|x|<|y|\ \Rightarrow x^2<\ y^2$
   c) $x^2<y^2\Rightarrow x<\ y$ d) $x^2<y^2\Rightarrow|x|<|y|$

a) If $y\leq 0$, it is always false; if $0\leq x$, is always true; in the rest of cases, depends.
b) and d) are always true.
c) If $y<0$, is always false; if $0<y$, is always true; if $y=0$,it is impossible.

5. For the sets $A\subset\mathbb{R}$ that are defined below, obtain the maximun and the minumun, if they exist, for $\alpha=-1$, $\alpha=0$ and $\alpha=1$

   a) $A=\{x:senx=\alpha\}$ b) $A=\{x:\cos x=\alpha\}$ c) $A=\{x:e^x\leq\alpha\}$
   d) $A=\{x:e^x\geq\alpha\}$ e) $A=\{x:\ln x\leq\alpha\}$ f) $A=\{x:\ln x\geq\alpha\}$

a) A has neither maximun nor minumun.
b) A has neither maximun nor minumun.
c) If $\alpha=-1$ or if $\alpha=0\Longrightarrow A$ has neither maximun nor minumun; if $\alpha=1\Longrightarrow A$ has no mínimun, but max(A)=0.
d) If $\alpha=-1$ or if $\alpha=0\Longrightarrow A$ has neither maximun nor minumun; if $\alpha=1\Longrightarrow A$ has no maximun, but min(A)=0.
e) If $\alpha=-1,\alpha=0$ or if $\alpha=1\Longrightarrow A$ has no minimun, but max(A)=$e^{-1},1,e$, respectively.
f) If $\alpha=-1,\alpha=0$ or if $\alpha=1\Longrightarrow A$ has no maximun, but min(A)=$e^{-1},1,e$, respectively.

6. (*)In $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ the following relation is defined: $(a,b) \leq (c,d)$ if and only if $a \leq c$ and $b \leq d$. Prove that "$\leq$" is a partial order relation.

   Let $A = \{(x,y) \in \mathbb{R}^2 \mid x+y \leq 1\}$, $B = \{(x,y) \in \mathbb{R}^2 | \ |x| \leq 1;\ |y| \leq 1\}$, $C = \{(x,y) \in \mathbb{R}^2 | \ 0 \leq y \leq 4-x^2\}$ $D = \{(x,y) \in \mathbb{R}^2 | \ x^2-9 \leq y \leq 0\}$ and $E = \left\{(x,y) \in \mathbb{R}^2 : |x| \leq y \leq 6-x^2\right\}$.

   Obtain for the previous sets, if they exist, the maximun and the minimun, the maximals and the minimals.

   a) maximals (A)=$\{(x,y) : x+y=1\}$.

   b) maximals (B)=Maximun (B)={(1,1)}; minimals (B)=Mínimun (B)={(-1,-1)}.

   c) maximals (C)={(x,y): $y = 4-x^2, 0 \leq x \leq 2$}; minimals(C)=Minimun(C)=$\{(-2,0)\}$.

   d) maximals (D)=Maximun (D)={(3,0)}; minimals (D)={(x,y): $y = x^2-9, -3 \leq x \leq 0$}.

   e) maximals (E)=$\left\{(x,y) \in \mathbb{R}^2 : y = 6-x^2, 0 \leq x \leq 2\right\}$; minimals (E)=$\left\{(x,y) \in \mathbb{R}^2 : y = -x, -2 \leq x \leq 0\right\}$.

7. (*)Let $f(x) = 1/x$ and $g(x) = x^2-1$.

   a) Find the domain and the range of those functions.
   b) Find $f(g(2))$ and $g(f(2))$.
   c) Find $f(g(x))$ and $g(f(x))$.

   a) Dom (f)=$(-\infty,0) \cup (0,\infty)$=Range (f). Dom (g)=$\mathbb{R}$, Range (g)=$[-1,\infty)$

   c) $f(g(x)) = \frac{1}{x^2-1}$ and $g(f(x)) = \frac{1}{x^2} - 1$.

8. (*)Find the domain and the range of the following functions:

   a)$f(x) = ln(senx)$ b)$g(x) = ln(sen^2x)$
   c) $h(x) = ln\sqrt{-x^2+4x-3}$

   a) Dom $(f) = \{x : senx > 0\} = \underset{k\in\mathbb{Z}}{\cup}(2k\pi,(2k+1)\pi)$; Range $(f) = (-\infty,0]$

   b) Dom $(f) = \{x : senx \neq 0\} = \underset{k\in\mathbb{Z}}{\cup}(k\pi,(k+1)\pi)$; Range $(f) = (-\infty,0]$

   c) Dom $(f) = \{x : -x^2+4x-3 > 0\} = \{x : x^2-4x+3 = (x-3)(x-1) < 0\} = (1,3)$;
   Range $(f) = (-\infty,0]$.

9. Review the graph of the functions:

   a)(*) $f(x) = x^2$ b)(*) $f(x) = e^x$ c)(*) $f(x) = lnx$ d) $f(x) = senx$

   In each case draw the graph of the following functions from the previous ones, interpreting geometrically the results.

   i) $g(x) = f(x+1)$ ii) $h(x) = -2f(x)$ iii) $p(x) = f(3x)$
   iv) $s(x) = f(x)+1$ v) $r(x) = |f(x)|$ vi) $m(x) = f(|x|)$

   i) Move the graph one unit to the left.

   ii) Strech the graph vertically $(2f(x))$ and, then, make a reflexion respect to the horizontal axis $(-2f(x))$.

   iii) Compress horizontally the graph.

   iv) Move vertically the graph one unit up.

   v) Mantain invariant the part of the graph that is above the horizontal axis, and obtain the symmetric part with respect to the horizontal axis of the part of the graph that is under that axis.

   vi) Mantain invariant the part of the graph that is on the right of the vertical axis, supress the left part of the graph and substituthe it for the symmetric of the part that is on the right of the vertical axis.

10. (*)Let $f,\ g : I \to \mathbb{R}$ be increasing funcitons. Discuss if the following statements are true or false

a) $f+g : I \to \mathbb{R}$ is an increasing function
b) $f \cdot g : I \to \mathbb{R}$ is an increasing function
c) $f-g : I \to \mathbb{R}$ is an increasing function if both functions are positive
d) $f-g : I \to \mathbb{R}$ is an increasing function if both functions are negative
a) Obvious.
b) If f, g possitive: obvious.
c) False.
d) False.

11. Let f, $g : \mathbb{R} \to \mathbb{R}$ be monotonic functions. Discuss when will $g \circ f$ be increasing or decreasing depending on the behaviour of f and g (four cases in total).

    a) gof will be increasing if both are increasing or decreasing.

    b) gof will be decreasing if one of them is increasing and the other is decreasing.

12. For each one of the following functions, for example f, find the intervals I, J for $f : I \to J$ to be bijective.

    a) $f(x) = x^2$; *b)* $g(x) = \ln|x|$; *c)* $h(x) = sen(x)$; *d)* $i(x) = e^{-x^2}$.

    c) $h : [-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi] \longrightarrow [-1, 1]$ *d)* $i : (-\infty, 0] \to (0, 1]$, $i : [0, \infty) \to (0, 1]$.

13 (*)Calculate the inverse function of the following functions:

$$f(x) = (x^3 - 5)^5, \quad g(x) = (\sqrt[3]{x} - 5)^5 \quad h(x) = \ln(\tfrac{x-1}{x-2}); i(x) = \tfrac{3x-1}{x-3};\ j(x) = \begin{cases} x+3 & -3 \le x \le 0 \\ -2x & 0 < x \le 3 \end{cases}$$

a) $f^{-1}(x) = \sqrt[3]{5 + \sqrt[5]{x}}$.

b) $g^{-1}(x) = 5 + x^{3/5} = 5 + (\sqrt[5]{x})^3$.

c) $h^{-1}(x) = \frac{2e^x - 1}{e^x - 1}$ d) $i^{-1}(x) = i(x)$ e) $j^{-1}(x) = \begin{cases} x-3 & 0 \le x \le 3 \\ -x/2 & -6 \le x < 0 \end{cases}$

14. Determine if the following functions are even, odd or neither of them:

    a) $f(x) = cos5x$ b)$g(x) = sen2x$ c)$h(x) = cos5xsen2x$ d)$k(x) = \dfrac{x^2}{x^2+1}$

    e)$l(x) = \dfrac{x^3}{x^4+1}$ f)$m(x) = \dfrac{x^3}{x^5+1}$ g) $n(x) = \dfrac{arctgx}{x}$

    a) Even.

    b) Odd.

    c) Odd.

    d) Even.

    e) Odd.

    f) Neither even nor odd.

    g) Even.

15. Let f be an even function and g an odd function. Prove that:

    $|g|$ is even; $f \circ g$ is even; $g \circ f$ is even;
    $f \cdot g$ is odd; g^k is even (if k is even); g^k is odd (if k is odd)

16. Determine which of the following functions are periodic and calculate its period.

    $f(x) = sen4x \quad g(x) = tg(\dfrac{x}{3}) \quad l(x) = sen(3x + 2)$

    a) $f(x)$ has period $2\pi/4$.

    b) $g(x)$ has period 3π.

    c) $l(x)$ has period $2\pi/3$.

17. Let f be any function and g a periodic function. Is it possible to state that $f \circ g$ and $g \circ f$ are periodic?

    Justify that $f(x) = \dfrac{tg^2 3x + ln(tg3x)}{1 + tg3x}$ is perodic.

    fog is a periodic function.

    On the other hand, gof is not necessarily periodic.