MATHEMATICS I: ORDERED SETS.

Definition 1: (X, $\leq$) is an ordered set when the relation $\leq$ satisfies the reflective, antisymmetric and transitive properties. That is:

i) reflective ($a \preceq a$);
ii) antisymmetric ($[a \preceq b$ y $b \preceq a] \Rightarrow a=b$);
iii) transitive $[a \preceq b$ y $b \preceq c\,] \Rightarrow a \preceq c$.

Definition 2: (X, $\leq$) is a total order if $\forall x, y \in X \Rightarrow [x \preceq y\ o\ y \preceq x]$.
Examples: (R, $\leq$) is a total order. $(R^2, \preceq_P)$ is a non-total or partial order defined by: $(x_1, y_1)\ \preceq_P (x_2, y_2) \Leftrightarrow [x_1 \leq x_2, y_1 \leq y_2]$.
Geometrically, this means that (x_1, y_1) is on the left and below with respect to (x_2, y_2). Therefore, (0,1) and (1,0) are not comparable.

Definition 3: if (X, $\leq$) is a fully ordered set, we define, for any $A \subset X, A \neq \emptyset$:
a) Maximum (A) = M$\Leftrightarrow[\forall a \in A \Rightarrow a \preceq M$ y $M \in A]$.
b) minimum (A) = m$\Leftrightarrow[\forall a \in A \Rightarrow m \preceq a$ y $m \in A]$.
Examples: if X=R, A finite and A = [0,1] have maximum, but neither A=[0,1) nor A=[0, ∞) have.

Remark 1: Similarly, it is possible to define maximum and minimum in (X, $\preceq$), a partially ordered set, but it is a little useful concept.
Example: $A=\{(x , y): x \geq 0, y \geq 0, x+ y \leq 1\}$ has no maximum. To solve this lack, we introduce:

Definition 4: If (X, $\preceq$) is a partially ordered set, for any $A \subset X$, $A \neq \emptyset$, we define:
a) Maximal elements (A)=$\{a \in A: \nexists a' \in A, a' \neq a$ y $a \preceq a'\}$.
b) Minimal elements (A)=$\{ a \in A: \nexists a' \in A, a' \neq a$ y $a' \preceq a\}$.
In this way, the set : $A=\{(x , y): x \geq 0, y \geq 0, x+ y \leq 1\}$ has
Maximal elements (A)=$\{(x, y) \in A: x+y=1\}$.

Remark 2: Observe that, if a maximum M exists, then it is the unique maximal element. The same is true for minimum and minimal elements.
But the opposite is not true: the set
$A=\{(x,y): x=0, 0 \leq y \leq 1\} \cup \{0<x<1, y=0\}$ has a unique maximal element, the point (0,1), but it has not a maximum.

Remark 3: a maximal element is also known as Pareto optimum. It is a basic concept in the economic language since the beginning of the 20th century.