WORKSHEET 2: Limits and Continuity

1. (*)Calculate

   a) $\lim\limits_{x\to 0} \dfrac{4x^3+2x^2-x}{5x^2+2x}$ b) $\lim\limits_{x\to 2} \dfrac{x^3-x^2-x-2}{x-2}$ c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

   d) $\lim\limits_{x\to \infty} \dfrac{x^2-\sqrt{x}}{\sqrt{x^3+3x^4}}$ e) $\lim\limits_{x\to \infty} \dfrac{sinx}{x}$ f) $\lim\limits_{x\to -\infty} \dfrac{x^2cosx+1}{x^2+1}$

   g) $\lim\limits_{x\to -\infty} \dfrac{3x^3+2x^2+x+2}{x^2-7x+1}$ h) $\lim\limits_{x\to -\infty} \dfrac{x^4-ax^3}{x^2+1}$ i) $\lim\limits_{x\to 0} \dfrac{x^4-x^3}{x^2+b}$

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2. Using that $\lim\limits_{x\to 0} \dfrac{sinx}{x} = 1$, calculate:

   a) $\lim\limits_{x\to 0} \dfrac{sin^2(2x)}{x^2}$ b) $\lim\limits_{x\to 1} \dfrac{sin(x^2-1)}{x-1}$

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3. Find the discontinuities,(if they exist) of the following functions:

   a)(*) $f(x) = \dfrac{|x-3|}{x-3}$ b) $f(x) = \begin{cases} x+\pi & if \quad x \le -\dfrac{\pi}{2} \\ \dfrac{xsinx}{1-cosx} & if \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}; \quad x \neq 0 \\ 1 & if \quad x = 0 \\ 0 & if \quad \dfrac{\pi}{2} \le x \end{cases}$

   c) $f(x) = \begin{cases} \dfrac{x+1}{-x} & if \quad x \le -1. \\ -1/2(1-x^{-2}) & if \quad -1 < x \le 1 \\ \dfrac{sin\pi x}{\pi} - 1 & if \quad 1 < x \end{cases}$ d) (*) $f(x) = \begin{cases} \dfrac{2x}{x+1} & if \quad x < -1. \\ e^{1/x} & if \quad -1 \le x < 0 \\ \pi & if \quad x = 0 \\ 1/x & if \quad 0 < x \end{cases}$

4. (*)Calcute the following limits:

   i) $\lim\limits_{x\to 1} \left\{ (x-1)arcsin(\dfrac{tg^4(x)}{1+tg^4(x)}) \right\}$

   ii) $\lim\limits_{x\to 2} \dfrac{1+h^2(x)}{|x-2|}$, with $h(x)$ a function with finite limit when $x \to 2$.

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5. (*)Calculate

   a) $\lim\limits_{x\to -3^+} \dfrac{x^2}{x^2-9}$ b) $\lim\limits_{x\to -3^-} \dfrac{x^2}{x^2-9}$ c) $\lim\limits_{x\to 0^+} \dfrac{2}{sinx}$ d) $\lim\limits_{x\to 0^-} (1-1/x)^{\frac{1}{x}}$ e) $\lim\limits_{x\to 0^-} \dfrac{x^2-2x}{x^3}$

6. Calculate all asymptotes of the following functions:

   (*) $f(x) = \dfrac{x^3}{x^2-1}$ $g(x) = \dfrac{x^2-1}{x}$ (*) $h(x) = \sqrt{x^2-1}$ (*) $m(x) = \dfrac{1}{lnx}$ (*) $n(x) = e^{1/x}$

7. Prove that every odd-degree polynomial has at least one root.

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8. (*)a) Use the intermediate value theorem to check that the following functions have a zero at the specified interval

   i) $f(x) = x^2-4x+3$ in $[2,4]$; ii) $g(x) = x^3+3x-2$ in $[0,1]$.

   b) Obtain using interval partitions and succesive applications of Bolzano, the zero with and error of ± 0.25.

9. (*)Check that the equations $x^4-11x+7=0$ and $2^x-4x=0$ have at least two solutions.

10. (*)Prove that the equation $x^7+3x+3=0$ has a unique solution. Determine the integer part of that solution.

11. Find the domain and the range of the functions:

$$\text{a)} f(x) = ln\left(\frac{(x^2-16)(x-1)}{x-3}\right) \quad \text{b)} g(x) = \sqrt{\frac{(x^2-16)(x-1)}{x-3}}$$

12. If f and g are continous functions in $[a,b]$ and $f(a) < g(a)$, $f(b) > g(b)$, prove that there exists a $x_0 \in (a,b)$ such that $f(x_0) = g(x_0)$

13. a) Let $f : [a,b] \to \mathbb{R}$, continuous, such that Range$(f) \subset [a,b]$. Prove that f has at least a fixed point.

 b) Also suppose that f is monotonic. Will exist an unique fixed point?

14. a) Prove using the Bolzano's theorem of zeroes, that the function $f(x) = x^3 - 5$
 has at least one fixed point in the interval $[0,n]$, for some $n \in \mathbb{N}$.

 b) Obtain, with an error of ± 0.25, a fixed point of f.
 c) Does a unique fixed point exist?

15. (*)Discuss in the following cases if the functions reach global and/or local extrema in the specified intervals:

 a) $f(x) = x^2 \quad x \in [-1,1]$ b) $f(x) = x^3 \quad x \in [-1,1]$

 c) $f(x) = sinx \quad x \in [0,\pi]$ d) $f(x) = -x^{\frac{1}{3}} \quad x \in [-1,1]$

16. In the previous problem, replace the interval given by $[0,\infty)$ or by $\mathbb{R}$ in each one of the functions.

17. Let $f(x) = arctg\left(\dfrac{tg^2x}{1+tg^4x}\right)$, $f : [a,b] \to \mathbb{R}$. Discuss, depending on the values of a and b, when f reaches maximun and minimun in $[a,b]$.

18. Explain why $f(x) = tgx$ has a maximun in $[0,\pi/4]$, but not in $[0,\pi]$.

19. (*)a) Let $C(x) = \dfrac{3x^2+x}{x-1} + 100$, be the total cost of production function, supposing $x \geq 7$.
 Check if it has oblique asymptote when $x \to \infty$.

 b) Consider the function $C_m(x) = \dfrac{C(x)}{x}$, that is, the average cost of production.
 Check that it has a horizontal asymptothe when $x \to \infty$.
 c) Is there any relationship between the oblique asymptote in part a) and the horizontal asymptote in part b?

20. (*)A banking entity offers a current account with the following conditions: the 250.000 fist euros non remunerated, the rest by a 7% of annual interest. Consider the following function: $i : [0,\infty) \to I\!R$ defined by $i(x)$="interes obtained in % when depositing some capital x and mantaining it during a year".

 i) Obtain $i(x)$.

 ii) Calculate $\lim_{x\to\infty} i(x)$.

 iii) Does any capital c exist such that $i(c) = 7$?.

 iv) From what capital is obtained at least a 6% of interest?

 v) Graph the function i.