Chapter 2

Limits and continuity of functions of one variable

2.1 Limits

To determine the behavior of a function f as x approaches a finite value c, we use the concept of limit. We say that the limit of f is L, and write $\lim_{x\to c} f(x) = L$, if the values of f approaches L when x gets closer to c.

Definition 2.1.1. (Limit when x approach a finite value c). We say that $\lim_{x\to c} f(x) = L$ if for any small positive ϵ, there is a positive δ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - c| < \delta$.

We can split the above definition in two parts, using one–sided limits.

Definition 2.1.2.

1. We say that L is the limit of f as x approaches c from the right, $\lim_{x\to c^+} f(x) = L$, if for any small positive ϵ, there is a positive δ such that

$$|f(x) - L| < \epsilon$$

   whenever $0 < x - c < \delta$.

2. We say that L is the limit of f as x approaches c from the left, $\lim_{x\to c^-} f(x) = L$, if for any small positive ϵ, there is a positive δ such that

$$|f(x) - L| < \epsilon$$

   whenever $0 < c - x < \delta$.

Theorem 2.1.3. $\lim_{x\to c} f(x) = L$ *if and only if*

$$\lim_{x\to c^+} f(x) = L \quad \textit{and} \quad \lim_{x\to c^-} f(x) = L.$$

We can also wonder about the behavior of the function f when x approaches $+\infty$ or $-\infty$.

Definition 2.1.4. (Limits when x approaches $\pm\infty$)

1. $\lim_{x\to+\infty} f(x) = L$ if for any small positive ϵ, there is a positive value of x, call it x_1, such that
$$|f(x) - L| < \epsilon$$
whenever $x > x_1$.

2. $\lim_{x\to-\infty} f(x) = L$ if for any small positive ϵ, there is a negative value of x, call it x_1, such that
$$|f(x) - L| < \epsilon$$
whenever $x < x_1$.

If the absolute values of a function become arbitrarily large as x approaches either a finite value c or $\pm\infty$, then the function has no finite limit L but will approach $-\infty$ or $+\infty$. It is possible to give the formal definitions. For example, we will say that $\lim_{x\to c} f(x) = +\infty$ if for any large positive number M, there is a positive δ such that

$$f(x) > M$$

whenever $0 < |x - c| < \delta$. Please, complete the remaining cases.

Note 2.1.5. Note that it could be $c \in D(f)$, so $f(c)$ is well defined, but $\lim_{x\to c} f(x)$ does not exits or $\lim_{x\to c} f(x) \neq f(c)$. Consider for instance the function f that is equal to 1 for $x \neq 0$, but $f(0) = 0$. Then clearly the limit of f at 0 is $1 \neq f(0)$.

Example 2.1.6. Consider the following limits.

1. $\lim\limits_{x\to 6} x^2 - 2x + 7 = 31$.

2. $\lim\limits_{x\to\pm\infty} x^2 - 2x + 7 = \infty$, because the leading term in the polynomial gets arbitrarily large.

3. $\lim\limits_{x\to+\infty} x^3 - x^2 = \infty$, because the leading term in the polynomial gets arbitrarily large for large values of x, but $\lim\limits_{x\to-\infty} x^3 - x^2 = -\infty$ because the leading term in the polynomial gets arbitrarily large in absolute value, and negative.

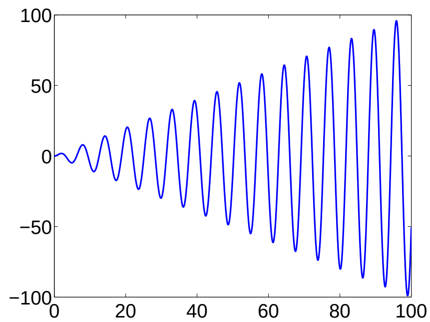
4. $\lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$, since for x arbitrarily large in absolute value, $1/x$ is arbitrarily small.

5. $\lim\limits_{x\to 0} \dfrac{1}{x}$ does not exists. Actually, the one–sided limits are:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

The right limit is infinity because $1/x$ becomes arbitrarily large when x is small and positive. The left limit is minus infinity because $1/x$ becomes arbitrarily large in absolute value and negative, when x is small and negative.

6. $\lim_{x\to+\infty} x \operatorname{sen} x$ does not exist. As x approaches infinity, $\operatorname{sen} x$ oscillates between 1 and -1. This means that $x \operatorname{sen} x$ changes sign infinitely often when x approaches infinity, whilst taking arbitrarily large absolute values. The graph is shown below.



7. Consider the function $f(x) = \begin{cases} x^2, & \text{if } x \le 0; \\ -x^2, & \text{if } 0 < x \le 1; \\ x, & \text{if } x > 1. \end{cases}$ $\lim_{x\to 0} f(x) = f(0) = 0$, but $\lim_{x\to 1} f(x)$ does not exist since the one–sided limits are different.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1,$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -x^2 = -1.$$

8. $\lim_{x\to 0} \frac{|x|}{x}$ does not exist, because the one–sided limits are different.

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1,$$
$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \qquad \text{(when } x \text{ is negative, } |x| = -x\text{).}$$

In the following, $\lim f(x)$ refer to the limit as x approaches $+\infty$, $-\infty$ or a real number c, but we never mix different type of limits.

2.1.1 Properties of limits

f and g are given functions and we suppose that all the limits below exist; $\lambda \in \mathbb{R}$ denotes an arbitrary scalar.

1. *Product by a scalar:* $\lim \lambda f(x) = \lambda \lim f(x)$.

2. *Sum:* $\lim(f(x) + g(x)) = \lim f(x) + \lim g(x)$.

3. *Product:* $\lim f(x)g(x) = (\lim f(x))(\lim g(x))$.

4. *Quotient:* If $\lim g(x) \neq 0$, then $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$.

Theorem 2.1.7 (Squeeze Theorem)**.** *Assume that the functions f, g and h are defined around the point c, except, maybe, for the point c itself, and satisfy the inequalities*

$$g(x) \leq f(x) \leq h(x).$$

Let $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. *Then*

$$\lim_{x\to c} f(x) = L.$$

Example 2.1.8. Show that $\lim\limits_{x\to 0} x \operatorname{sen}\left(\dfrac{1}{x}\right) = 0$.

Solution: We use the theorem above with $g(x) = -|x|$ and $h(x) = |x|$. Notice that for every $x \neq 0$, $-1 \leq \operatorname{sen}(1/x) \leq 1$ thus, when $x > 0$

$$-x \leq x \operatorname{sen}(1/x) \leq x,$$

and when $x < 0$

$$x \leq x \operatorname{sen}(1/x) \leq -x.$$

These inequalities mean that $-|x| \leq x \operatorname{sen}(1/x) \leq |x|$. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

we can use the theorem above to conclude that $\lim_{x\to 0} x \operatorname{sen}\frac{1}{x} = 0$.

2.1.2 Techniques for evaluating $\lim \dfrac{f(x)}{g(x)}$

1. Use the property of the quotient of limits, if possible.

2. If $\lim f(x) = 0$ and $\lim g(x) = 0$, try the following:

   (a) Factor $f(x)$ and $g(x)$ and reduce $\frac{f(x)}{g(x)}$ to lowest terms.

   (b) If $f(x)$ or $g(x)$ involves a square root, then multiply both $f(x)$ and $g(x)$ by the conjugate of the square root.

   Example 2.1.9.

$$\lim_{x\to 3} \frac{x^2-9}{x+3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x+3} = \lim_{x\to 3}(x-3) = 0.$$

$$\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{1-\sqrt{1+x}}{x}\left(\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right) = \lim_{x\to 0} \frac{-x}{x(1+\sqrt{1+x})} = \lim_{x\to 0} \frac{-1}{1+\sqrt{1+x}} = -\frac{1}{2}.$$

3. If $f(x) \neq 0$ and $\lim g(x) = 0$, then either $\lim \frac{f(x)}{g(x)}$ does not exist or $\lim \frac{f(x)}{g(x)} = +\infty$ or $-\infty$.

4. If x approaches $+\infty$ or $-\infty$, divide the numerator and denominator by the highest power of x in any term of the denominator.

   Example 2.1.10.

$$\lim_{x\to\infty} \frac{x^3 - 2x}{-x^4 + 2} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{2}{x^3}}{-1 + \frac{2}{x^4}} = \frac{0 - 0}{-1 + 0} = 0.$$

2.1.3 Exponential limits

Let the limit

$$\lim_{x\to c} [f(x)]^{g(x)}$$

be an indetermination. This happens if

- $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$ (1^∞).
- $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ (0^0).
- $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 0$ (∞^0).

Noting that

$$\lim_{x\to c} [f(x)]^{g(x)} = \lim_{x\to c} e^{g(x)\ln f(x)} = e^{\lim_{x\to c} g(x)\ln f(x)},$$

all cases are reduced to the indetermination $0 \cdot \infty$, since we have to compute the limit

$$\lim_{x\to c} g(x) \ln f(x).$$

In the first indetermination, 1^∞, it often helps to use the identity

$$\lim_{x\to c} g(x) \ln f(x) = \lim_{x\to c} g(x)(f(x) - 1).$$

since when x is close to 0, $\ln(1 + x) \approx x$, or, $\ln x \approx x - 1$ when x is close to 1.

Example 2.1.11. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1+\frac{1}{x}\right)} = e^{x\frac{1}{x}} = e.$

Example 2.1.12. Let $a, b > 0$. Calculate $\lim_{x\to\infty} \left(\frac{1+ax}{2+bx}\right)^x$.

If $a > b$, then the basis function tends to $a/b > 1$, thus the limit is ∞. If $a < b$, then the basis function tends to $a/b < 1$, thus the limit is 0. When $a = b$

$$\lim_{x\to\infty} \left(\frac{1 + ax}{2 + ax}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{1+ax}{2+ax} - 1\right)} = e^{\lim_{x\to\infty} \frac{-x}{2+ax}} = e^{-1/a}.$$

2.1.4 Remarkable limit

Recall that

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1.$$

Example 2.1.13. Evaluate the following limits:

1. $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \frac{1}{\cos x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1.$

2. $\lim_{x\to 0} \frac{\operatorname{sen} 3x}{x} \overset{\{z=3x\}}{=} \lim_{z\to 0} \frac{\operatorname{sen} z}{\frac{z}{3}} = 3 \lim_{z\to 0} \frac{\operatorname{sen} z}{z} = 3.$

2.2 Asymptotes

An *asymptote* is a line that the graph of a function approaches more and more closely until the distance between the curve and the line almost vanishes.

Definition 2.2.1. Let f be a function

1. The line $x = c$ is a vertical asymptote of f if $\lim_{x\to c^+} |f(x)| = \infty$ or $\lim_{x\to c^-} |f(x)| = \infty$.
2. The line $y = b$ is a horizontal asymptote of f if $\lim_{x\to +\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$.
3. The line $y = ax + b$ is an oblique asymptote of f if
   (a) $\lim\limits_{x\to +\infty} \frac{f(x)}{x} = a$ and $\lim\limits_{x\to +\infty} (f(x) - ax) = b$, or
   (b) $\lim\limits_{x\to -\infty} \frac{f(x)}{x} = a$ and $\lim\limits_{x\to -\infty} (f(x) - ax) = b$.

Notice that a horizontal asymptote is a particular case of oblique asymptote with $a = 0$.

Example 2.2.2. Determine the asymptotes of $f(x) = \frac{(1+x)^4}{(1-x)^4}$.

Solution: Since the denominator vanishes at $x = 1$, the domain of f is $\mathbb{R} - \{1\}$. Let us check that $x = 1$ is a vertical asymptote of f:

$$\lim_{x\to 1^\pm} \frac{(1+x)^4}{(1-x)^4} = +\infty$$

On the other hand

$$\lim_{x\to +\infty} \frac{(1+x)^4}{(1-x)^4} = \lim_{x\to +\infty} \frac{(1/x+1)^4}{(1/x-1)^4} = 1$$

hence $y = 1$ is a horizontal asymptote at $+\infty$. In the same way, $y = 1$ is a horizontal asymptote at $-\infty$. There is no other oblique asymptotes.

Example 2.2.3. Determine the asymptotes of $f(x) = \frac{3x^3 - 2}{x^2}$.

Solution: The domain of f is $\mathbb{R} - \{0\}$. Let us check that $x = 0$ is a vertical asymptote of f.

$$\lim_{x\to 0^\pm} \frac{3x^3 - 2}{x^2} = \lim_{x\to 0^\pm} (3x - \frac{2}{x^2}) = \lim_{x\to 0^\pm} 3x - \lim_{x\to 0^\pm} \frac{2}{x^2} = -\infty.$$

Thus, $x = 0$ is a vertical asymptote of f. On the other hand

$$\lim_{x\to \pm\infty} \frac{3x^3 - 2}{x^2} = \lim_{x\to \pm\infty} (3x - \frac{2}{x^2}) = \pm\infty$$

thus, there is no horizontal asymptote. Let us study now oblique asymptotes:

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{3x^3-2}{x^3} = \lim_{x\to\pm\infty}\left(3-\frac{2}{x^3}\right) = 3,$$
$$b = \lim_{x\to\pm\infty}(f(x)-3x) = \lim_{x\to\pm\infty}\left(\frac{3x^3-2}{x^2}-3x\right) = \lim_{x\to\pm\infty}\left(-\frac{2}{x^2}\right) = 0.$$

We conclude that $y = 3x$ is an oblique asymptote both at $+\infty$ and $-\infty$.

2.3 Continuity

The easiest limits to evaluate are those involving continuous functions. Intuitively, a function is continuous if one can draw its graph without lifting the pencil from the paper.

Definition 2.3.1. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at c if $c \in D(f)$ and

$$\lim_{x\to c} f(x) = f(c).$$

Hence, f is *discontinuous at* c if either $f(c)$ is undefined or $\lim_{x\to c} f(x)$ does not exist or $\lim_{x\to c} f(x) \neq f(c)$. Moreover, we can define one-sided continuity of f at c,

Definition 2.3.2. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *right continuous* at c, if $c \in D(f)$ and

$$\lim_{x\to c^+} f(x) = f(c).$$

f is *left continuous* at c, if $c \in D(f)$ and

$$\lim_{x\to c^-} f(x) = f(c).$$

Obviously, a function f is continuous at c when is both, right and left continuous at c.

2.3.1 Properties of continuous functions

Suppose that the functions f and g are both continuous at c. Then the following functions are also continuous at c.

1. *Sum.* $f + g$.
2. *Product by a scalar.* λf, $\lambda \in \mathbb{R}$.
3. *Product.* fg.
4. *Quotient.* f/g, whenever $g(c) \neq 0$.

2.3.2 Limit and continuity of the composite function

Theorem 2.3.3. *Let f, g be functions from $\mathbb{R}$ to $\mathbb{R}$ and let $c \in \mathbb{R}$. If g is continuous at L and $\lim_{x\to c} f(x) = L$, then*

$$\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x)) = g(L).$$

If the function f is continuous at c, then, calling $L = f(c)$ the result above becomes:

Corollary 2.3.4. *Let f be a continuous function at c and g continuous on $f(c)$. Then , the composite function $g \circ f$ is also continuous at c.*

Example 2.3.5. Compute the following limits:

- $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \ln(1+x)^{1/x} = \ln\left(\lim\limits_{x\to 0}(1+x)^{1/x}\right) = \ln e = 1.$

  Note that the function $\ln(\cdot)$ is continuous at e, then we can apply 2.3.4.

- $\lim\limits_{x\to 0} \dfrac{a^x-1}{x} = \lim\limits_{z\to 0} \dfrac{z}{\frac{\ln(1+z)}{\ln a}} = \ln a \left(\lim\limits_{z\to 0} \dfrac{z}{\ln(1+z)}\right) = \ln a.$

  We have used the substitution $z = a^x - 1$, so that $x = \ln(1+z)/\ln a$, and we have used the value of the limit computed before.

2.3.3 Continuity of elementary functions

A function is called *elementary* if it can be obtained by means of a finite number of arithmetic operations and superpositions involving basic elementary functions. The functions $y = C =$ constant, $y = x^a$, $y = a^x$, $y = \ln x$, $y = e^x$, $y = \operatorname{sen} x$, $y = \cos x$, $y = \tan x$, $y = \arctan x$ are examples of elementary functions. *Elementary functions are continuous in their domain.*

Example 2.3.6.

1. The function $f(x) = \sqrt{4-x^2}$ is the composition of the functions $y = 4 - x^2$ and $f(y) = y^{1/2}$, which are elementary, thus f is continuous in its domain, that is, in $D = [-2, +2]$.

2. The function $g(x) = \frac{1}{\sqrt{4-x^2}}$ is the composition of function f above and function $g(y) = 1/y$, thus it is elementary and continuous in its domain, $D(g) = (-2, +2)$.

2.3.4 Continuity of the inverse function

A one-to-one function (also named bijectve) does not have to be continuous. For example, the following function

$$f(x) = \begin{cases} 1, & \text{si } x = 0; \\ x, & \text{si } 0 < x < 1; \\ 0, & \text{si } x = 1. \end{cases}$$

is bijective considering that its domain and image are the interval $[0, 1]$.

It can be shown that neither $f(x)$ is continuous nor $f^{-1}(x)$, which coincidentally happens to be the same f.

This will not be the case should the function $f(x)$ be continuous, as the following theorem proves:

Theorem 2.3.7. *Let $f : I \longrightarrow J$ be continuous and bijective Then:*

a) f is strictly increasing (or decreasing), and

b) The inverse f^{-1} is a continuous function as well.

OBSERVATION: Obviously, f^{-1} is also strictly increasing (o decreasing), depending on f having the same nature as well.

Example 2.3.8. Prove that $\lim_{x\to 1} \arctan\left(\dfrac{x^2+x-2}{3x^2-3x}\right) = \dfrac{\pi}{4}$.

SOLUTION: The function *arctan* $= \tan^{-1}$ is continuous from what we just have seen above. Then applying theorem 2.3.3:

$$\begin{aligned}\lim_{x\to 1} \arctan\left(\frac{x^2+x-2}{3x^2-3x}\right) &= \arctan\left(\lim_{x\to 1}\frac{x^2+x-2}{3x^2-3x}\right)\\ &= \arctan\left(\lim_{x\to 1}\frac{(x-1)(x+2)}{3x(x-1)}\right)\\ &= \arctan\left(\lim_{x\to 1}\frac{x+2}{3x}\right)\\ &= \arctan 1\\ &= \frac{\pi}{4}.\end{aligned}$$

2.3.5 Continuity theorems

Continuous functions have interesting properties. We shall say that a function is continuous on the *closed* interval $[a, b]$ if it is continuous at every point $x \in (a, b)$, is right-continuous at a and left-continuous at b.

Theorem 2.3.9 (Bolzano's Theorem)**.** *If f is continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$, then there exists some $c \in (a, b)$ such that $f(c) = 0$.*

Example 2.3.10. Show that the equation $x^3 + x - 1 = 0$ admits a solution, and find it with an error less than 0.1.

SOLUTION: With $f(x) = x^3 + x - 1$ the problem is to show that there exists c such that $f(c) = 0$. We want to apply Bolzano's Theorem. First, f is continuous in $\mathbb{R}$. Second, we identify a suitable interval $I = [a, b]$. Notice that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ thus, there is a solution $c \in (0, 1)$.

Now, to find an approximate value for c, we use a method of *interval–halving* as follows: consider the interval $[0.5, 1]$; $f(0.5) = 1/8 + 1/2 - 1 < 0$ and $f(1) > 0$, thus $c \in (0.5, 1)$. Choose now the interval $[0.5, 0.75]$; $f(0.5) < 0$ and $f(0.75) = 27/64 + 3/4 - 1 > 0$ thus,

$c \in (0.5, 0.75)$. Let now the interval $[0.625, 0.75]$; $f(0.625) \approx -0.13$ and $f(0.74) > 0$ thus, $c \in (0.625, 0.75)$. The solution is approximately $c = 0.6875$ with a maximum error of 0.0625.

The previous theorem, known as Bolzano's Theorem can be generalized for every intermediate value between $f(a)$ and $f(b)$, since it is proved in the following theorem.

Theorem 2.3.11 (Intermediate Value Theorem)**.** *Let f be a continuous function on the closed interval $[a, b]$. Then, for any intermediate real number k between $f(a)$ and $f(b)$, there is at least a number $x_k \in [a, b]$ satisfying $f(x_k) = k$.*

Notice: An intermediate value means any real number k with $f(a) < k < f(b)$ or $f(b) < k < f(a)$.

Proof. Consider the function $g(x) = f(x) - k$. Then, $g(a) < 0 < g(b)$ or $g(b) < 0 < g(a)$. Applying Bolzano's Theorem to the function g, there is $x_k \in [a, b]$ such that $g(x_k) = 0$. Similarly, there exists a $x_k \in [a, b]$ such that $f(x_k) = k$. $\square$

The following result is very useful when you are trying to find the image of a continuous function.

Corollary 2.3.12. *Let f be a continuous, non-constant function defined on any interval I (not necessarily closed or bounded). Then, $J = Im(f)$ is also an interval.*

Notice: J does not always satisfy the same properties of the interval I.

Example 1: $f(x) = 1/x$ is continuous on the bounded interval $I = (0, 1]$, but $J = Im(f) = [1, \infty)$ is not bounded.

Example 2: $f(x) = 1/x$ is continuous on the closed interval $I = [1, \infty)$, but $J = Im(f) = (0, 1]$ is not closed.

Nevertheless, if the interval I is compact, ie: it is closed and bounded, then J is also compact.

This last result is called Weierstrass' Theorem, and it is the most important of chapter 2.

Theorem 2.3.13 (Weierstrass' Theorem)**.** *If f is continuous in $[a, b]$, then there exist points $c, d \in [a, b]$ such that*

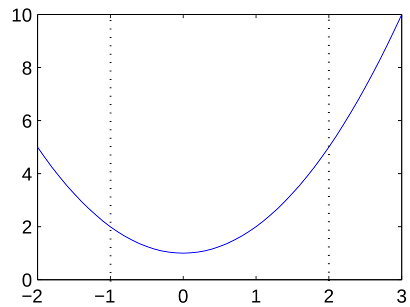
$$f(c) \leq f(x) \leq f(d)$$

for every $x \in [a, b]$.

The theorem asserts that a continuous function attains over a closed interval a minimum ($m = f(c)$) and a maximum value ($M = f(d)$). The point c is called a *global minimum* of f on $[a, b]$ and d is called a *global maximum* of f on $[a, b]$.

Example 2.3.14. Show that the function $f(x) = x^2 + 1$ attains over the closed interval $[-1, 2]$ a minimum and a maximum value.

Solution: The graph of f is shown below.



We can see that f is continuous in $[-1,2]$, actually f is continuous in $\mathbb{R}$, and f attains the minimum value at $x=0$, $f(0)=1$, and the maximum value at $x=2$, $f(2)=5$.

Example 2.3.15. The assumptions in the Theorem of Weierstrass are essential.

- The interval is not closed, or not bounded.
  - Take $I=(0,1]$ and $f(x)=1/x$; f is continuous in I, but it does not have global maximum.
  - Take $I=[0,\infty)$ and $f(x)=1/(1+x)$; f is continuous in I, but it does not have global minimum, since $\lim_{x\to\infty} f(x)=0$, but $f(x)>0$ is strictly positive for every $x\in I$.
- The function is not continuous. Take $I=[0,1]$ and $f(x)=\begin{cases} x, & \text{if } 0\le x<1; \\ 0, & \text{if } x=1. \end{cases}$; f has a global minimum at $x=0$, but there is no global maximum since $\lim_{x\to 1} f(x)=1$ but $f(x)<1$ for every $x\in I$.

2.3.6 Fixed points

Definition 2.3.16. Let $f: I\to J$. We say that the point $x^*\in I$ is a fixed point of f when $f(x^*)=x^*$.

Graphically, x^* is a fixed point when the graph of $f(x)$ intersects the main diagonal $y=x$.

Note 2.3.17. If we consider the function $g(x)=f(x)-x$, then it is obvious that the fixed points of $f(x)$ corresponds to the zeroes of $g(x)$.

Example 2.3.18. Let's consider the function $f(x)=x^2$ on $[0,1]$. Then, obviously, the fixed points are 0 and 1.

Example 2.3.19. Let $f:[a,b]\to$ be continuous and such that $[a,b]\subset \mathrm{Im}(f)$ Then, f has at least a fixed point. Observe that there are points x_a and x_b such that $f(x_a)=a$ and $f(x_b)=b$.

If we consider the interval limited by the points x_a and x_b, we observe that $g(x)=f(x)-x$ satisfies that $g(x_a)\le 0, g(x_b)\ge 0$, so $g(x)$ has a zero and then $f(x)$ has a fixed point.

Note 2.3.20. It is important to note that, though an increasing or decreasing function on an interval I has an unique root, or none, this result for fixed points remains true only for decreasing functions, but not for increasing ones. Consider the example 1.

2.3.7 Equilibrium of a market

We know that if f and g are defined on an interval $[a, b]$, both functions are continuous, the first one is increasing, the second one decreasing and $f(a) < g(a), f(b) > g(b)$, then, there exists a single point x_0 such that $f(x_0) = g(x_0)$.

To prove that equality, you have only to observe that the function $g(x) - f(x)$ (or $f(x) - g(x)$) has a zero. If we consider x to be the quantity of a certain commodity, and $f(x), g(x)$, respectively, the price at which this quantity is offered and demanded, we call:

1. x_0 the quantity of equilibrium for that market,
2. $f(x_0) = p_0 = g(x_0)$ the price of equilibrium; and
3. the pair (x_0, p_0) the equilibrium of that market.

The situation is a bit more complicated if we consider the interval $[0, \infty)$ instead of the interval $[a, b]$. In this case, it is reasonable to assume for the offer function that $f(0) = 0$, i.e., that, at a price of zero the offer is null, and that if $x \to \infty$, then $f(x) \to \infty$, as the constraints to the production provoke that the price is higher and higher.

For the demand function is also reasonable to argue that if $x \to \infty$, then $g(x) \to 0$, as the market will be saturated with such a big production.

With respect to the demand near zero, we have two possibilities:

i) $\lim_{x \to 0+} g(x)$ is finite; or

ii) $\lim_{x \to 0+} g(x) = \infty$.

In the case i), if we consider the interval $[a, b] = [0, M]$, where M satisfies that $f(M) > g(M)$, it is clear that we have an equilibrium for that market.

The case ii) is a bit more complicated. In this case, we consider the interval $[a, b] = [m, M]$ where $m > 0$ satisfies that $f(m) < g(m)$ and M satisfies, again, that $f(M) > g(M)$; then, it is clear again that we have an equilibrium for that market.

In both cases the equilibrium is unique.