Universidad Carlos III de Madrid

Exercise	1	2	3	4	Total
Points					

Department of Economics **Introduction to Mathematics Extra-Final Exam** **June 18th 2024**

Exam time: 1 hour and 40 minutes.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = \ln(1 + e^{2x})$. Then:**
- (a) find every asymptote of $f(x)$.
- (b) find the increasing/decreasing intervals and the range of $f(x)$.
- (c) find the intervals where the function $f(x)$ is convex or concave and sketch its graph.

Hint for part b): $\ln(a) - b = \ln a - \ln(e^b)$.

0.3 points part a); 0.3 points part b); 0.4 points part c)

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a) First of all, since $f(x)$ is continuous in its domain, $\mathbb{R}$, we only need to look for asymptotes at $\pm\infty$.

As, $\lim_{x\to-\infty} \ln(1+e^{2x}) = \ln 1 = 0$ then $y = 0$ is a horizontal asymptote at $-\infty$.

Now at, ∞ :

$\lim_{x\to\infty} \dfrac{\ln(1+e^{2x})}{x} = \dfrac{\infty}{\infty} =$ (using L'Hopital) $= \lim_{x\to\infty} \dfrac{2e^{2x}}{1+e^{2x}} = \lim_{x\to\infty} \dfrac{2}{e^{-2x}+1} = 2$; and:

$\lim_{x\to\infty} \ln(1+e^{2x}) - 2x = \lim_{x\to\infty} \ln(1+e^{2x}) - \ln e^{2x} =$

$= \lim_{x\to\infty} \ln \dfrac{1+e^{2x}}{e^{2x}} = \lim_{x\to\infty} \ln(e^{-2x}+1) = \ln 1 = 0$, then $y = 2x$ is an oblique asymptote at ∞.

b) To find the intervals where the function $f(x)$ is increasing or decreasing, we calculate the derived function and study its sign:

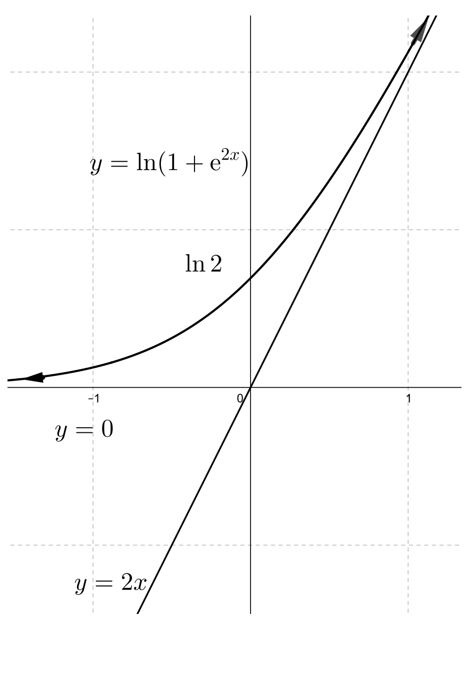
$f'(x) = \dfrac{2e^{2x}}{1+e^{2x}} > 0$, so f is increasing in its whole domain $\mathbb{R}$. Furthermore, since $f(x)$ is continuous and increasing in its domain, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$, using the Intermediate Value theorem for continuous functions, we can deduce that the range of the function is $(0, \infty)$.

c) To find the intervals where the function is convex or concave we calculate the second order derived function:

$$f''(x) = \left(\frac{2e^{2x}}{1+e^{2x}}\right)' = \left(\frac{2+2e^{2x}-2}{1+e^{2x}}\right)' = \left(2 - \frac{2}{1+e^{2x}}\right)' = \frac{4e^{2x}}{(1+e^{2x})^2} > 0,$$

so the function $f(x)$ is convex for the entire real line.

Therefore, the approximate graph of the function can be seen in the figure below.



(2) **Given the equation** $32\sqrt{2+x} - y - y^3 = 62$, **it is asked:**

(a) Prove that the equation defines an implicit fucntion $y = f(x)$ in a neighbourhood of the point $x = 2, y = 1$.

(b) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 2$.

(c) approximately sketch the graph of the function f near the point $x = 2, y = 1$. Calculate the approximate value of $f(1,9)$ using the tangent line. Compare the obtained result with the exact value of $f(1,9)$, knowing that $f''(2) < 0$.

0.2 points part a); 0.4 points part b); 0.4 points part c)

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a) Considering the function $F(x,y) = 32\sqrt{2+x} - y - y^3$, must be satisfied:

i) First of all, $F(2,1) = 64 - 1 - 1 = 62$; ii) Secondly, the function is continuously differentiable in a neighbourhood of the point; and iii) Finally, $\dfrac{\partial F}{\partial y} = -1 - 3y^2 \Longrightarrow \dfrac{\partial F}{\partial y}(2,1) = -4 \neq 0$.

Then, the equation implicitly defines the function $y = f(x)$ in a neighborhood of the point $x = 2,\ y = 1$.

b) To start with, we calculate the first-order derivative of the equation:

$$\frac{32}{2\sqrt{2+x}} - y' - 3y^2y' = 0$$

evaluating at $x = 2,\ y(2) = 1$ we obtain: $8 = 4y'(2) \Longrightarrow f'(2) = 2$. Then the equation of the tangent line is: $y = P_1(x) = 1 + 2(x-2)$.

Analogously, we calculate the second-order derivative of the equation:

$$\frac{(-1/2)16}{(2+x)^{3/2}} - y'' - 6y(y')^2 - 3y^2y'' = 0$$

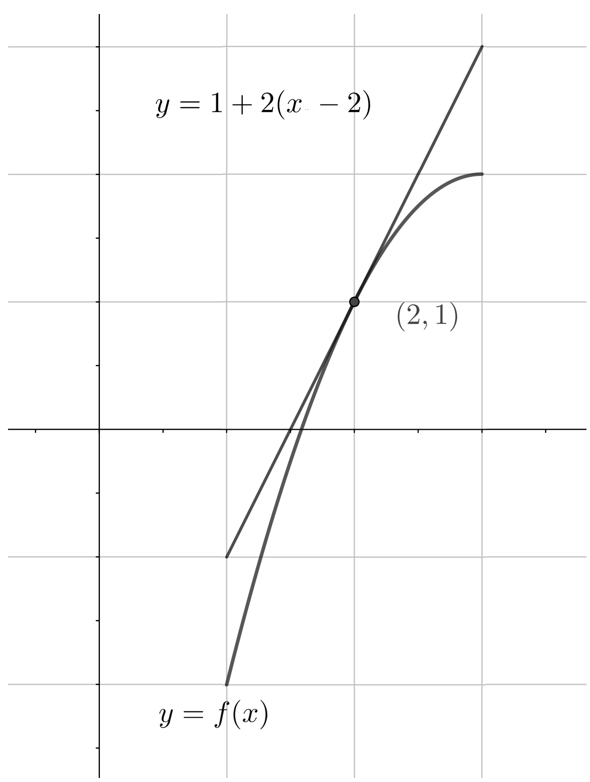
evaluating at $x = 2,\ y(2) = 1, y'(2) = 2$ we obtain:

$$\frac{-8}{8} - 4y''(2) - 6\cdot 2^2 = 0 \Longrightarrow -25 = 4y''(2) \Longrightarrow y''(2) = -\frac{25}{4}.$$

Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 1 + 2(x-2) - \dfrac{25}{8}(x-2)^2$.

c) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 2$, will be as you can see in the figure underneath.

Moreover, using the tangent line at $x = 2$, we obtain: $f(1.9) \approx 1 + 2(1.9 - 2) = 0.8$ and we also know that $f(1.9) < 0.8$, since $f(x)$ is concave near $x = 2$.



(3) **Let** $C(x) = C_0 + 2x + x^2$ **be the cost function and** $p(x) = A - 2x$ **be the inverse demand function of a monopolistic firm, where** $A, C_0 > 0$ **. It is asked:**

(a) Calculate the value of A, C_0 such that the firm's profits are maximized at the level of production $x^* = 8$.

(b) Calculate the value of A, C_0 such that the firm's minimum average cost is obtained at the level of production $x^* = 8$.

0.5 points part a); 0.5 points part b).

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a) The profit function is

$B(x) = (A - 2x)x - (C_0 + 2x + x^2) = -3x^2 + (A - 2)x - C_0$

Now, we calculate the first and second order derivatives of B:

$B'(x) = -6x + A - 2,$ and $B''(x) = -6 < 0$

So we know that B has only one critical point at $x^* = \dfrac{A-2}{6}$ and since B is a concave function, the critical point is a strictly global maximizer.

Hence, $x^* = 8 = \dfrac{A-2}{6} \Longrightarrow A = 50;$ and C_0 can take any real value.

b) The average cost function is $\dfrac{C(x)}{x} = x + 2 + \dfrac{C_0}{x},$ and its first order derivative is

$\left(\dfrac{C(x)}{x}\right)' = 1 - \dfrac{C_0}{x^2} = 0 \Longleftrightarrow x^2 = C_0.$

Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{2C_0}{x^3} > 0,$ the average cost function is convex and the critical point is a strictly global minimizer.

Therefore, $x^{**} = 8 \Longrightarrow C_0 = 64;$ and A can take any real value.

(4) **Given the function $f(x) = x^3 - 6x$, it is asked:**

(a) state The Bolzano's Theorem (or zero existence Theorem) for a function g defined in $[a,b]$.

(b) find the zeros or roots of $f(x)$ and the intervals where $f(x)$ takes positive and negative signs.

(c) determine the values of a, b so that the function $f : [a,b] \to \mathbb{R}$ fulfills the hypothesis of the theorem. Determine the values of a, b so that the function $f : [a,b] \to \mathbb{R}$ does not fulfill the hypothesis, but the thesis or conclusion of the theorem is satisfied.

0.2 points part a); 0.4 points part b); 0.4 points part c)

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a) See the class's notes.

b) Obviously, $f(x) = x(x - \sqrt{6})(x + \sqrt{6})$ so, the roots of f are 0, $\sqrt{6}$ and $-\sqrt{6}$.

Considering that $-3 < -\sqrt{6} < -1 < 0 < 1 < \sqrt{6} < 3$, $f(-3) < 0 < f(-1)$ and $f(1) < 0 < f(3)$ we obtain:

i) $f(x) < 0$ when $x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$.

ii) $f(x) > 0$ when $x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$.

c) Based on the previous data, the function $f : [a,b] \to \mathbb{R}$ satisfies the hypotheses of Bolzano's Theorem when one of the following four cases occurs:

i) $a < -\sqrt{6} < b < 0$

ii) $a < -\sqrt{6}, \sqrt{6} < b$

iii) $-\sqrt{6} < a < 0 < b < \sqrt{6}$

iv)$0 < a < \sqrt{6} < b$.

On the other hand, the function $f : [a,b] \to \mathbb{R}$ It satisfies the thesis of Bolzano's theorem, although not the hypotheses, when one of the following two cases occurs:

i) $a < -\sqrt{6}, 0 < b < \sqrt{6}$

ii) $-\sqrt{6} < a < 0, \sqrt{6} < b$

The graph of the function can help to understand this result.