Universidad Carlos III de Madrid

Exercise	1	2	3	4
Points				

Department of Economics **Introduction to Mathematics Final Exam** **January 9th 2025**

Exam time: 1 hour and 35 minutes.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = e^{6x-x^2}$. Then:**

(a) find the asymptotes and the increasing/decreasing intervals of $f(x)$.

(b) find the local and global extreme points and the range of $f(x)$. Draw the graph of the function.

(c) consider the function $f_1(x)$ restricted to the interval where $f(x)$ it is increasing. Draw the graph of the inverse function of $f_1(x)$.

0.4 points part a); 0.4 points part b); 0.2 points part c).

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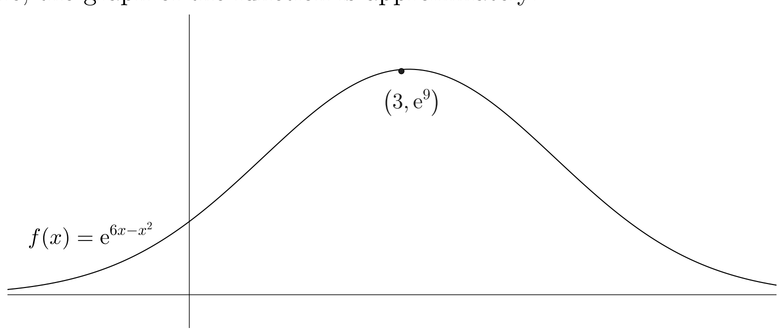
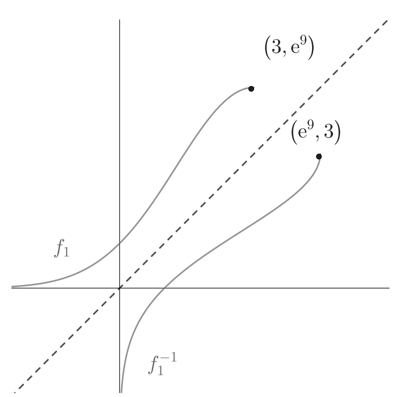
a) The domain of $f(x)$ is $\mathbb{R}$. Since f is continuous in its domain, we only need to study its asymptotes on $-\infty$ and ∞. Observing that $\lim\limits_{x\longrightarrow\pm\infty}(6x-x^2) = -\infty$, we can deduce that $y = 0$ is the horizontal asymptote of the function on $\pm\infty$.
On the other hand, as $f'(x) = e^{6x-x^2}(6-2x)$, we obtain that $x = 3$ is the only critical point of f and we deduce that f is increasing on $(-\infty, 3]$, because $f'(x) > 0$ on $(-\infty, 3)$. Analogously, f is decreasing on $[3, \infty)$.

b) From the above we know that $x = 3$ is a local and global maximizer. Moreover, given that there is no local minimizer, there cannot be a global minimizer either.
Further more, since f is continuous on $\mathbb{R}$, monotonic in the intervals found and $\lim\limits_{x\longrightarrow-\infty} f(x) = \lim\limits_{x\longrightarrow\infty} f(x) = 0$, using the Intermediate Value Theorem it is deduced that the range of f is $(0, f(3)] = (0, e^9]$. Therefore, the graph of the function is approximately:



c) As we can notice, f_1 is increasing in $(-\infty, 3]$, $f_1(3) = e^9$, $f_1(x)$ has an horizontal asymptote which equation is $y = 0$ at $-\infty$ and its range is $(0, e^9]$. Then, its inverse function is define and it is increasing in $(0, e^9]$, it takes the value 3 at e^9, and has a vertical asymptote with equation $x = 0$. The graph of the function f_1 and its inverse are approximately:



(2) **Given the implicit function $y = f(x)$, defined by the equation $x^2 - x + e^{-y} = 1$ in a neighbourhood of the point $x = 0$, $y = 0$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 0$.

(b) approximately sketch the graph of the function $f(x)$ and its inverse $f^{-1}(x)$ near the point $x = 0$.

(c) find the analytical expression of $f^{-1}(x)$.

(*Hint for part (c): If $y = f(x)$ satisfies the equation $F(x, y) = C$, then $y = f^{-1}(x)$ will satisfy $F(y, x) = C$*)

0.4 points part a); 0.4 points part b); 0.2 points part c).

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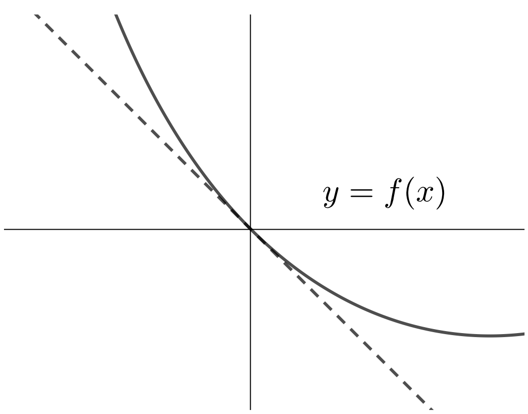
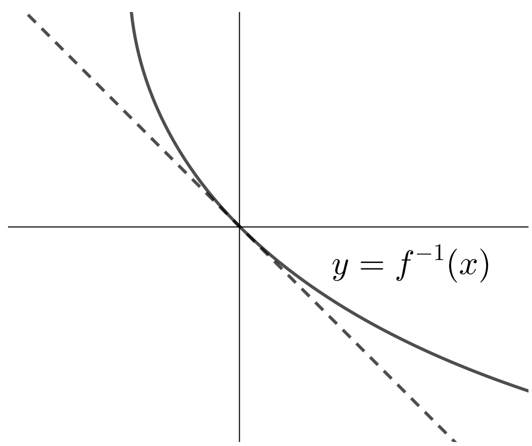
a) First of all, we notice that (0,0) is a solution of the equation. Now, we calculate the first order-derivative of the equation with respect to x at the point $x = 0$, $y(0) = 0$: $2x - 1 - y'e^{-y} = 0$ to obtain $y'(0) = f'(0) = -1$.
Then the equation of the tangent line is: $y = P_1(x) = -x$.
Analogously, we calculate the second-order derivative of the equation: $2 + (-y'' + (y')^2)e^{-y} = 0$ evaluating at $x = 0$, $y(0) = 0$, $y'(0) = -1$ we obtain: $y''(0) = f''(0) = 3$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = -x + \frac{3}{2}x^2$

b) Using the second-order Taylor Polynomial to approximate the graph of the function f, near the point $x = 0$, and the symmetry of its inverse function with respect to the principal diagonal ($y = x$) we can sketch both graphs and they can be seen in the figures bellow:



(A) Gráfica de $f(x)$



(B) Gráfica de $f^{-1}(x)$

c) As $y = f^{-1}(x)$ satisfies the equation $y^2 - y + e^{-x} = 1 \iff y^2 - y + e^{-x} - 1 = 0$ we can deduced that:

$$y = \frac{1 \pm \sqrt{1 - 4(e^{-x} - 1)}}{2} = \frac{1 \pm \sqrt{5 - 4e^{-x}}}{2}.$$

¿Which sign should we choose? One possibility it is to notice that the point (0,0) solves the equation.

Thus, $0 = \dfrac{1 \pm \sqrt{5 - 4e^{-0}}}{2}$, then $y = \dfrac{1 - \sqrt{5 - 4e^{-x}}}{2}$.

Other possibility, it is to know that $f^{-1}(x)$ is decreasing. Since e^{-x} is decreasing, the function $\sqrt{5 - 4e^{-x}}$ is increasing, hence the need to choose the negative sign.

(3) **Let $C(x) = 16 + 5x + 4x\sqrt{x}$ be the cost function of a monopolistic firm and $p(x) = 35 - \sqrt{x}$ be the inverse demand function. It is asked:**

(a) calculate the production $\hat{x}$, such that the firm's profit is maximized.

(b) find the production x^* where the derivative of the average cost function is zero. Prove that this function is **NOT** convex.

(c) is x^* the global minimizer of the average cost function?

(*Hint for part (c): sketch approximately the graph of the function* $\dfrac{C(x)}{x}$)

0.4 points part a); 0.4 points part b); 0.2 points part c).

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a) Fist of all, we calulate the profit function: $B(x) = (35-\sqrt{x})x-(16+5x+4x\sqrt{x}) = -5x\sqrt{x}+30x-16$. Then we calculate its first and second order derivatives:

$B'(x) = -\dfrac{15}{2}\sqrt{x} + 30; \qquad B''(x) = -\dfrac{15}{4\sqrt{x}} < 0.$

We observe that B has only one critical point at $\hat{x} = \left(2 \cdot \dfrac{30}{15}\right)^2 = 16$ and, since B is a concave function, this critical point is the only global maximizer.

b) The average cost function is

$\dfrac{C(x)}{x} = \dfrac{16}{x} + 5 + 4\sqrt{x}$, with $x \neq 0$,

We calculate its first and second order derivatives:

$$\left(\frac{C(x)}{x}\right)' = -\frac{16}{x^2} + \frac{4}{2\sqrt{x}}; \qquad \left(\frac{C(x)}{x}\right)'' = \frac{32}{x^3} - \frac{1}{x\sqrt{x}}.$$

We observe that the average cost function has only one critical point at

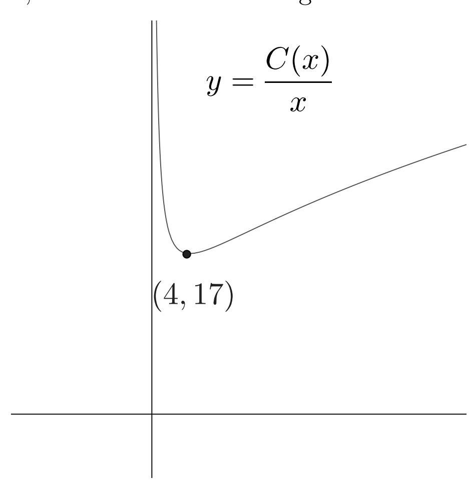
$-\dfrac{16}{x^2} + \dfrac{4}{2\sqrt{x}} = 0 \Longleftrightarrow x^2 = 8\sqrt{x} \iff (\sqrt{x})^3 = 8 \iff \sqrt{x} = 2 \iff x^* = 4,$

with $\left(\dfrac{C(4)}{4}\right)'' = \dfrac{32}{64} - \dfrac{1}{8} > 0$ and then $x^* = 4$ is a local minimizer.

However, taking $x = 100$, $\left(\dfrac{C(100)}{100}\right)'' = \dfrac{32}{1000000} - \dfrac{1}{1000} < 0$, then the function is not convex and we cannot ensure that the critical point is the global minimizer for the average cost function.

c) Now, studying the monotonicity of the function from the sign of its first order derivative we observe that: $\left(\dfrac{C(x)}{x}\right)' < 0$ if $0 < x < 4$; and $\left(\dfrac{C(x)}{x}\right)' > 0$ when $x > 4$.

Hence, $\dfrac{C(x)}{x}$ is decreasing in $(0,4]$ and increasing in $[4,\infty)$. Therefore, the critical point is the only global minimizer of $\dfrac{C(x)}{x}$, as it is shown in the figure:



(4) **Given the function** $f(x) = \begin{cases} x^2 - 2x + a^2 & x < 2 \\ x^2 - 7x + 12 & x \geqslant 2 \end{cases}$ **Then:**

(a) state Bolzano's Zero Theorem for the function f defined on the interval $[1, K]$, where $K > 2$. Determine the values of a and K for the function $f(x)$ so the hypothesis (or initial conditions) of the theorem is satisfied.

(b) state Lagrange's Mean Value Theorem for a function f defined on $[-1, 2]$. Find the value of a such that the hypothesis of the theorem is satisfied.

For the found values of a, calculate the point or points c where the thesis (or conclusion) of the theorem is satisfied.

0.5 points part a); 0.5 points part b).

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a) The hypothesis is that f is continuous in $[1, K]$ and also $f(1) \cdot f(K) < 0$.
The thesis or conclusion is that there exist a point $c \in (1, K)$ such that $f(c) = 0$.
First of all, we need that f is continuous at $x = 2$. Since $\lim_{x \longrightarrow 2^-} f(x) = a^2$, $f(2) = \lim_{x \longrightarrow 2^+} f(x) = 2$, we can deduce that the function is continuous on $[0, K]$ when $a = \pm\sqrt{2}$.
Secondly, supposing f continuous, we obtain $f(1) = -1 + a^2 = -1 + (\pm\sqrt{2})^2 = 1 > 0$, then the condition $f(1) \cdot f(K) < 0$ is satisfied when $f(K) < 0$.
On the other hand, we have $x^2 - 7x + 12 = (x - 3)(x - 4)$, then $f(K) < 0$ if $3 < K < 4$.
Finally, the hypothesis of Bolzano's Theorem is satisfied if:$a = \pm\sqrt{2}$, and $3 < K < 4$.

b) The hypothesis of the theorem is that f is continuous on $[-1, 2]$ and derivable in $(-1, 2)$.
The thesis or conclusion is that there is a point $c \in (-1, 2)$ such that $\dfrac{f(2) - f(-1)}{3} = f'(c)$.
We have already seen that the function is continuous on $[-1, 2]$ when $a = \pm\sqrt{2}$.
But now, we don't need the function to be derivable at $x = 2$.
Since $f(2) - f(-1) = -3 \Longrightarrow \frac{-3}{3} = -1 = f'(c) = (2c - 2)$, it is satisfied if
$2c - 2 = -1 \Longrightarrow c = 1/2 \in (-1, 2)$.