February 14, 2020

CHAPTER 2: LIMITS AND CONTINUITY OF FUNCTIONS IN EUCLIDEAN SPACE

1. Scalar product in $\mathbb{R}^n$

Definition 1.1. Given $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n$,we define their **scalar product** as

$$x \cdot y = \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Example 1.2. $(2, 1, 3) \cdot (-1, 0, 2) = -2 + 6 = 4$

Remark 1.3. $x \cdot y = y \cdot x$.

Definition 1.4. Given $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ we define its **norm** as

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}$$

Example 1.5. Example: $\|(-1, 0, 3)\| = \sqrt{10}$

Remark 1.6. The following are some interpretations of the norm.

- The norm $\|x\|$ is the distance from x to the origin.
- We may also interpret $\|x\|$ as the length of the vector x.
- The norm $\|x - y\|$ is the distance between x and y.

Remark 1.7. Let θ be the angle between u and v. Then,

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

2. 1. The Euclidean space $\mathbb{R}^n$

Definition 2.1. Given $p \in \mathbb{R}^n$ and $r > 0$ we define the **open ball** of center p and radius r as the set

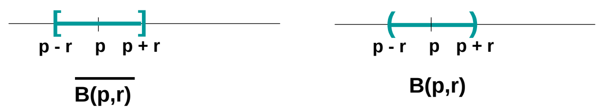
$$B(p, r) = \{y \in \mathbb{R}^n : \|p - y\| < r\}$$

and the **closed ball** of center p and radius r as the set

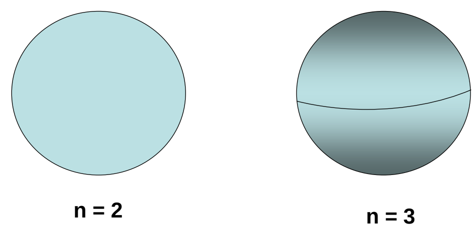
$$\overline{B(p, r)} = \{y \in \mathbb{R}^n : \|p - y\| \leq r\}$$

Remark 2.2.

- Recall that $\|p - y\|$ is distance from p to y.
- For $n = 1$, we have that $B(p, r) = (p - r, p + r)$ and $\overline{B(p, r)} = [p - r, p + r]$.



- For $n = 2, 3$ the closed balls are



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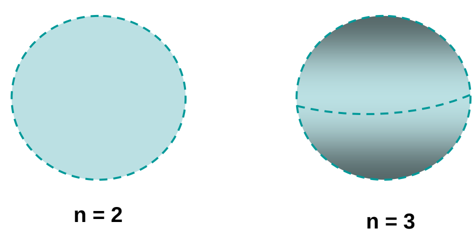
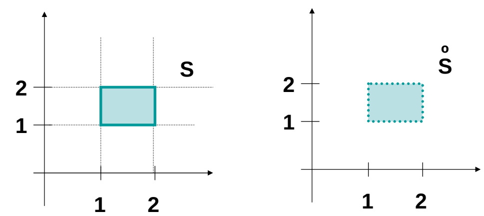


Definition 2.3. Let $S \subset \mathbb{R}^n$. We say that $p \in \mathbb{R}^n$ is **interior** to S if there is some $r > 0$ such that $B(p, r) \subset S$.

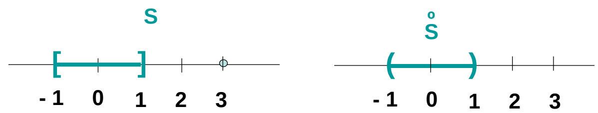
Notation: $\overset{\circ}{S}$ is set of interior points of S.

Remark 2.4. Note that $\overset{\circ}{S} \subset S$ because $p \in B(p, r)$ for any $r > 0$.

Example 2.5. Consider $S \subset \mathbb{R}^2, S = [1, 2] \times [1, 2]$. Then, $\overset{\circ}{S} = (1, 2) \times (1, 2)$.

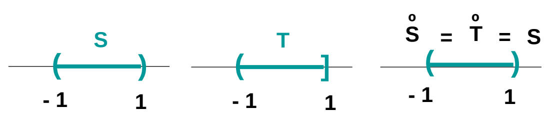


Example 2.6. Consider $S = [-1, 1] \cup \{3\} \subset \mathbb{R}$. Then, $\overset{\circ}{S} = (-1, 1)$.

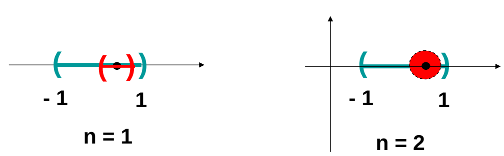


Definition 2.7. A subset $S \subset \mathbb{R}^n$ is **open** if $S = \overset{\circ}{S}$

Example 2.8. In $\mathbb{R}$, the set $S = (-1, 1)$ is open, $T = (-1, 1]$ is not.



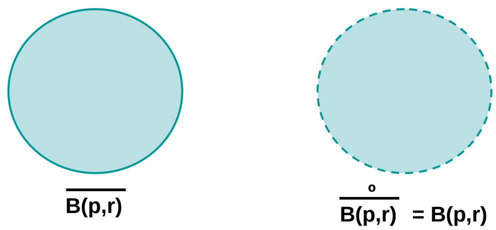
Example 2.9. The set $S = \{(x, 0) : -1 < x < 1\}$ is not open in $\mathbb{R}^2$.



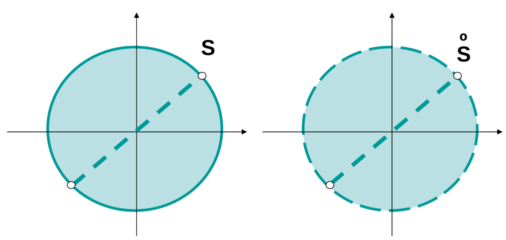
You should compare this with the previous example

Example 2.10. The open ball $B(p, r)$ is an open set.

Example 2.11. The closed ball $\overline{B(p,r)}$ is not an open set, because $\overset{\circ}{\overline{B(p,r)}} = B(p,r)$.



Example 2.12. Consider the set $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\overset{\circ}{S} = \{(x, y) \in R^2 : x^2 + y^2 < 1, x \neq y\}$. So, S is not open.

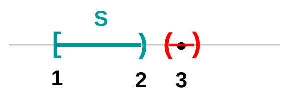


Proposition 2.13. $\overset{\circ}{S}$ is the largest open set contained in S. (That is $\overset{\circ}{S}$ is open, $\overset{\circ}{S} \subset S$ and if $A \subset S$ is open, then $A \subset \overset{\circ}{S}$).

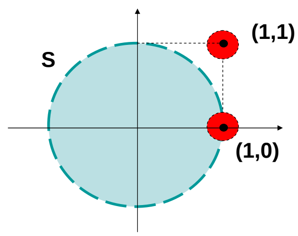
Definition 2.14. Let $S \subset \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is in the **closure** of S if for any $r > 0$ we have that $B(p, r) \cap S \neq \emptyset$.

Notation: $\bar{S}$ is the set of points in the closure of S.

Example 2.15. Consider the set $S = [1, 2) \subset \mathbb{R}$. Then, the points $1, 2 \in \bar{S}$. But, $3 \notin \bar{S}$.

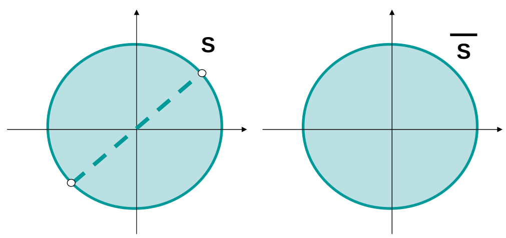


Example 2.16. Consider the set $S = B\left((0,0),1\right) \subset \mathbb{R}^2$. Then, the point $(1,0) \in \bar{S}$. But, the point $(1,1) \notin \bar{S}$.



Example 2.17. Let $S = [0,1]$, $T = (0,1)$. Then, $\bar{S} = \bar{T} = [0,1]$.

Example 2.18. Let $S = \{(x,y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\bar{S} = \overline{B\left((0,0),1\right)}$.



Example 2.19. $\overline{B(p,r)}$ is the closure of the open unit ball $B(p,r)$.

Remark 2.20. $S \subset \bar{S}$.

Definition 2.21. A set $F \subset \mathbb{R}^n$ is **closed** if $F = \bar{F}$.

Proposition 2.22. A set $F \subset \mathbb{R}^n$ is closed if and only if $\mathbb{R}^n \setminus F$ is open.

Example 2.23. The set $[1,2] \subset \mathbb{R}$ is closed. But, the set $[1,2) \subset \mathbb{R}$ is not.

Example 2.24. The set $\overline{B(p,r)}$is closed. But, the set $B(p,r)$ is not.

Example 2.25. The set $S = \{(x,y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$is not closed.

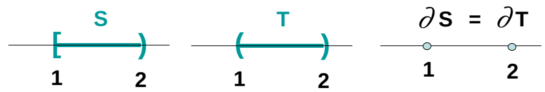
Proposition 2.26. The closure $\bar{S}$ of S is the smallest closed set that contains S. (That is $\bar{S}$ is closed, $S \subset \bar{S}$ and if F is another closed set that contains S, then $\bar{S} \subset F$).

Definition 2.27. Let $S \subset \mathbb{R}^n$, we say that $p \in \mathbb{R}^n$ is a **boundary point** of S if for any positive radius $r > 0$, we have that,

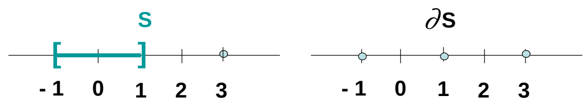
(1) $B(p,r) \cap S \neq \emptyset$.
(2) $B(p,r) \cap (\mathbb{R}^n \setminus S) \neq \emptyset$.

Notation: The set of boundary points of S is denoted by ∂S.

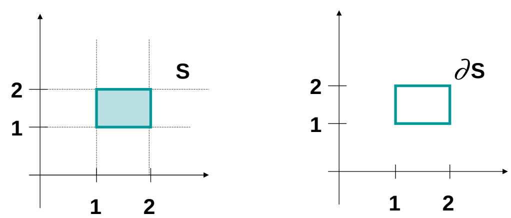
Example 2.28. Let $S = [1, 2)$, $T = (1, 2)$. Then, $\partial S = \partial T = \{1, 2\}$.



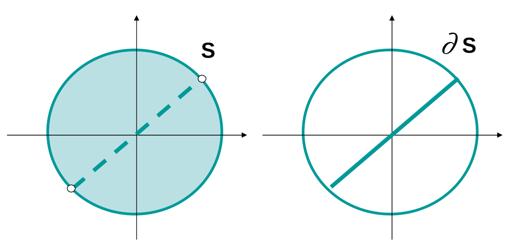
Example 2.29. Let $S = [-1, 1] \cup \{3\} \subset \mathbb{R}$. Then, $\partial S = \{-1, 1, 3\}$.



Example 2.30. Let $S \subset \mathbb{R}^2, S = [1, 2] \times [1, 2]$. Then, ∂S is



Example 2.31. $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$. Then, $\partial S = \{(x, y) : x^2 + y^2 = 1\} \bigcup \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x = y\}$.



The above concepts are related in the following Proposition.

Proposition 2.32. Let $S \subset \mathbb{R}^n$, then

(1) $\overset{\circ}{S} = S \setminus \partial S$
(2) $\bar{S} = S \cup \partial S$
(3) $\partial S = \overline{S} \cap \overline{\mathbb{R}^n \setminus S}$.
(4) S is closed $\Leftrightarrow S = \overline{S} \Leftrightarrow \partial S \subset S$
(5) S is open $\Leftrightarrow S = \overset{\circ}{S} \Leftrightarrow S \cap \partial S = \emptyset$.

Proposition 2.33.

(1) The finite intersection of open (closed) sets is also open (closed).
(2) The finite union of open (closed) sets is also open (closed).

Definition 2.34. A set $S \subset \mathbb{R}^n$ is **bounded** if there is some $R > 0$ such that $S \subset B(0, R)$.

Example 2.35. The straight line $V = \{(x, y, z) \in \mathbb{R}^3 : x - y = 0, z = 0\}$ is not a bounded set.

Example 2.36. The ball $B(p, R)$ of center p and radius R is bounded.

Definition 2.37. A subset $S \subset \mathbb{R}^n$ is **compact** if S is closed and bounded.

Example 2.38. $S = \{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$ is not compact (bounded, but not closed).

Example 2.39. $B(p, R)$ is not compact (bounded, but not closed).

Example 2.40. $\overline{B(p, R)}$ is compact.

Example 2.41. $(0, 1]$ is not compact. $[0, 1]$ is compact.

Example 2.42. $[0, 1] \times [0, 1]$ is compact.

Definition 2.43. A subset $S \subset \mathbb{R}^n$ is **convex** if for any $x, y \in S$ and $\lambda \in [0, 1]$ we have that $\lambda \cdot x + (1 - \lambda) \cdot y \in S$.

Example 2.44. Let A a matrix of order $n \times m$ and let $b \in \mathbb{R}^m$. We define

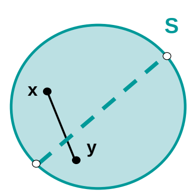
$$S = \{x \in \mathbb{R}^n : Ax = b\}$$

as the set of solutions of the linear system of equations $Ax = b$. Let $x, y \in S$, be two solutions of this linear system of equations. Then, we have that $Ax = Ay = b$. If we now take any $0 \leq t \leq 1$ (indeed any $t \in \mathbb{R}$) then

$$A(tx + (1 - t)y) = tAx + (1 - t)Ay = tb + (1 - t)b = b$$

that is, $tx + (1 - t)y \in S$ so the set of solutions of a linear system of equations is a convex set.

Example 2.45. $\{(x, y) \in R^2 : x^2 + y^2 \leq 1, x \neq y\}$ is not a convex set.



3. Function of several variables

We study now functions $f : \mathbb{R}^n \to \mathbb{R}$

Example 3.1.

- $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

  $$f(x, y) = x + y - 1$$

  also

  $$f(x, y) = x \sin y$$

- $f : \mathbb{R}^3 \to \mathbb{R}$ defined by
$$f(x, y, z) = x^2 + y^2 + \sqrt{1 + z^2}$$
also
$$f(x, y, z) = z \exp x^2 + y^2$$
- $f : \mathbb{R}^4 \to \mathbb{R}$ defined by
$$f(x, y, z, t) = \sin x + y + z \exp t.$$

Occasionally, we will consider functions $f : \mathbb{R}^n \to \mathbb{R}^m$ like, for example, $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by
$$f(x, y, z) = (x \exp y + \sin z, x^2 + y^2 - z^2)$$
But, if we write $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$ with
$$f_1(x, y, z) = x \exp y + \sin z, \quad f_2(x, y, z) = x^2 + y^2 - z^2$$
Then, $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$. So, we may just focus on functions $f : \mathbb{R}^n \to \mathbb{R}$.

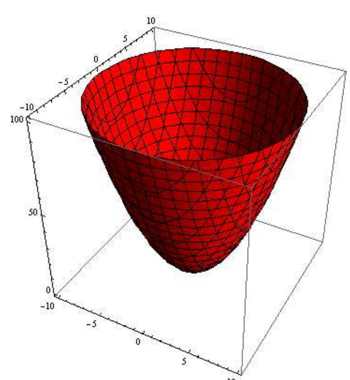
Remark 3.2. When we write
$$f(x, y, z) = \frac{\sqrt{x + y + 1}}{x - 1}$$
it is understood that $x \neq 1$. That is the expression of f defines implicitly the domain of the function. For example, for the above function we need that $x+y+1 \geq 0$ and $x \neq 1$. So, we assume implicitely that the domain of $f(x, y, z) = \frac{\sqrt{x+y+1}}{x-1}$ is the set
$$D = \{(x, y) \in \mathbb{R}^2 : x + y \geq -1, x \neq 1\}$$

Usually we will write $f : D \subset \mathbb{R}^n \to \mathbb{R}$ to make explicit the domain of f.

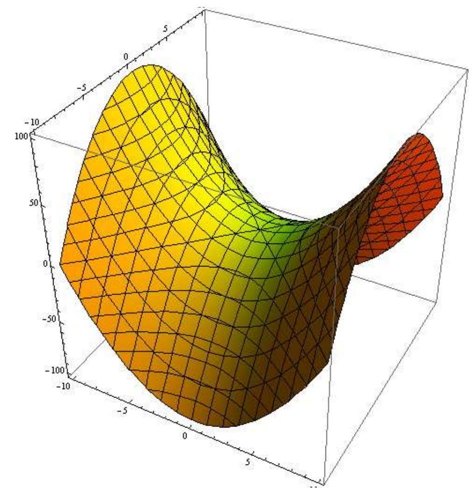
Definition 3.3. Given $f : D \subset \mathbb{R}^n \to \mathbb{R}$ we define the **graph** of f as
$$G(f) = \{(x, y) \in \mathbb{R}^{n+1} : y = f(x), x \in D\}$$

Remark that the graph can be drawn only for $n = 1, 2$.

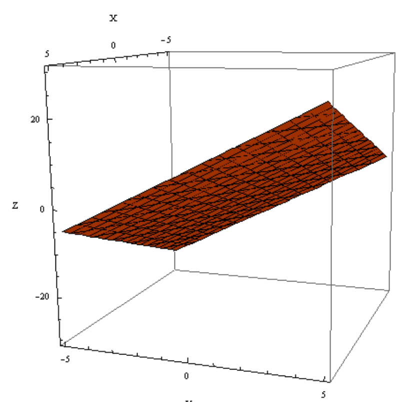
Example 3.4. The graph of $f(x, y) = x^2 + y^2$ is



Example 3.5. The graph of $f(x, y) = x^2 - y^2$ is



Example 3.6. The graph of $f(x, y) = 2x + 3y$ is



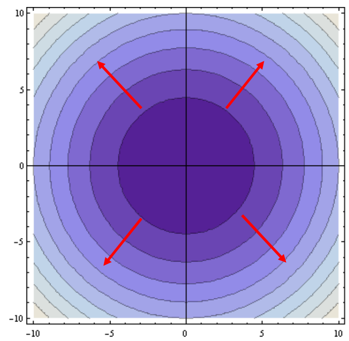
4. Level curves and level surfaces

Definition 4.1. Given $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{R}$ we define the **level surface** of f as the set

$$C_k = \{x \in D : f(x) = k\}.$$

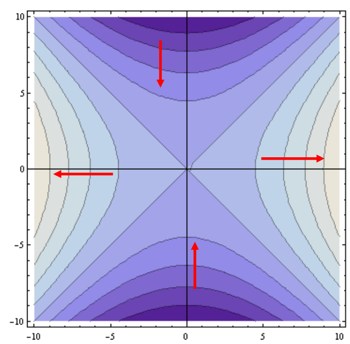
If $n = 2$, the level surface is called a **level curve**.

Example 4.2. The level curves of $f(x, y) = x^2 + y^2$ are



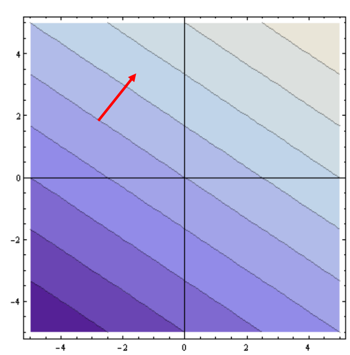
The arrows point in the direction in which the function f grows.

Example 4.3. The level curves of $f(x,y)=x^2-y^2$ are



The arrows point in the direction in which the function f grows.

Example 4.4. The level curves of $f(x,y)=2x+3y$ are



The arrows point in the direction in which the function f grows.

5. Limits and continuity

Definition 5.1. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and let $L \in \mathbb{R}$, $p \in \mathbb{R}^n$. We say that

$$\lim_{x \to p} f(x) = L$$

if given $\varepsilon > 0$ there is some $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever $0 < \|x - p\| < \delta$.

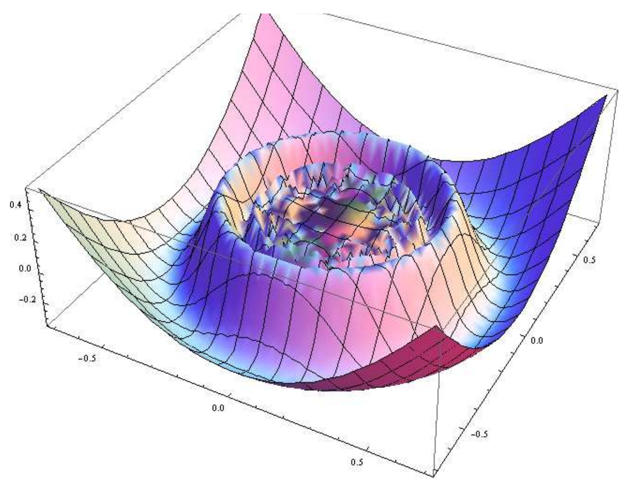
This is the natural generalization of the concept of limit for one-variable functions to functions of several variables, once we remark that the distance $|\ |$ in $\mathbb{R}$ is replaced by the distance $\|\ \|$ in $\mathbb{R}^n$). Note that interpretation is the same, i.e., $|x - y|$ is the distance from x to y in $\mathbb{R}$ and $\|x - y\|$ is the distance from x to y in $\mathbb{R}^n$.

Proposition 5.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ and suppose there are two numbers, L_1 and L_2 that satisfy the above definition of limit. That is, $L_1 = \lim_{x \to p} f(x)$ and $L_2 = \lim_{x \to p} f(x)$. Then, $L_1 = L_2$

Remark 5.3. The calculus of limits with several variables is more complicated than the calculus of limits with one variable.

Example 5.4. Consider the function

$$f(x, y) = \begin{cases} (x^2 + y^2) \cos(\frac{1}{x^2 + y^2}) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$



We will show that

$$\lim_{(x,y) \to (0,0)} f(x, y) = 0$$

In the above definition of limit we take $L = 0$, $p = (0, 0)$. We have to show that given $\varepsilon > 0$ there is some $\delta > 0$ such that

$$|f(x, y)| < \varepsilon$$

whenever $0 < \|(x,y)\| < \delta$, where

$$\|(x,y)\| = \sqrt{x^2+y^2}$$

So, fix $\varepsilon > 0$ and take $\delta = \sqrt{\varepsilon} > 0$. Suppose that

$$0 < \|(x,y)\| = \sqrt{x^2+y^2} < \delta = \sqrt{\varepsilon}$$

then,

$$x^2 + y^2 < \varepsilon$$

and $(x,y) \neq (0,0)$ so,

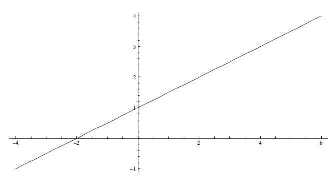
$$|f(x,y)| = \left|(x^2+y^2)\cos(\frac{1}{x^2+y^2})\right| < \varepsilon \left|\cos(\frac{1}{x^2+y^2})\right| \leq \varepsilon$$

where we have used that $|\cos(z)| \leq 1$ for any $z \in \mathbb{R}$. It follows that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

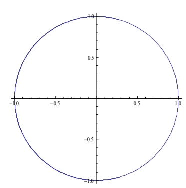
Remark 5.5. The above definition of limit needs to be modified to take care of the case in which there are no points $x \in D$ (where D is the domain of f) such that $0 < ||p-x|| < \delta$ For example, what is $\lim_{x\to -1} \ln(x)$? To avoid formal complication, we will only study $\lim_{x\to p} f(x)$ for the cases in which the set $\{x \in D : 0 < ||p-x|| < \delta\} \neq \emptyset$, for every $\delta > 0$

Definition 5.6. : A map $\sigma(t) : (a,b) \to \mathbb{R}^n$ is called a **curve** in $\mathbb{R}^n$.

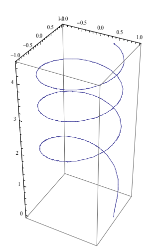
Example 5.7. $\sigma(t) = (2t, t+1), t \in \mathbb{R}$.



Example 5.8. $\sigma(t) = (cos(t), \sin(t)), t \in \mathbb{R}$.



Example 5.9. $\sigma(t) = (cos(t), \sin(t), \sqrt{t}), \sigma : \mathbb{R} \to \ \mathbb{R}^3$.



Proposition 5.10. Let $p \in D \subset \mathbb{R}^n$ and $f : D \subset \mathbb{R}^n \to \mathbb{R}$. Consider a curve $\sigma : [-\varepsilon, \varepsilon] \to D$ such that $\sigma(0) = p$ $\sigma(t) \neq p$ whenever $t \neq 0$ and $\lim_{t\to 0} \sigma(t) = p$. Suppose, $\lim_{x\to p} f(x) = L$. Then,

$$\lim_{t\to 0} f(\sigma(t)) = L$$

Remark 5.11. The previous proposition is useful to prove that a limit does not exist or to compute that value of the limit if we know in advance that the limit exists.

But, it cannot be used to prove that a limit exists since one of the hypotheses of the proposition is that the limit exists.

Remark 5.12. Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$. Let $p = (a, b)$ consider the following particular curves

$$\sigma_1(t) = (a + t, b)$$

$$\sigma_2(t) = (a, b + t)$$

Note that

$$\lim_{t\to 0} \sigma_i(t) = (a, b) \quad i = 1, 2$$

so, if

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

then, we must also have

$$\lim_{x\to a} f(x, b) = \lim_{y\to b} f(a, y) = L$$

Remark 5.13. **Iterated limits**

Suppose that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and that the following one-dimensional limits

$$\lim_{x\to a} f(x, y)$$

$$\lim_{y\to b} f(x, y)$$

exist for (x, y) in a ball $B((a, b), R)$. Define the functions

$$g_1(y) = \lim_{x\to a} f(x, y)$$

$$g_2(x) = \lim_{y\to b} f(x, y)$$

Then,

$$\lim_{x\to a} \left(\lim_{y\to b} f(x, y) \right) = \lim_{x\to a} g_2(x) = L$$

$$\lim_{y\to b} \left(\lim_{x\to a} f(x, y) \right) = \lim_{y\to b} g_1(y) = L$$

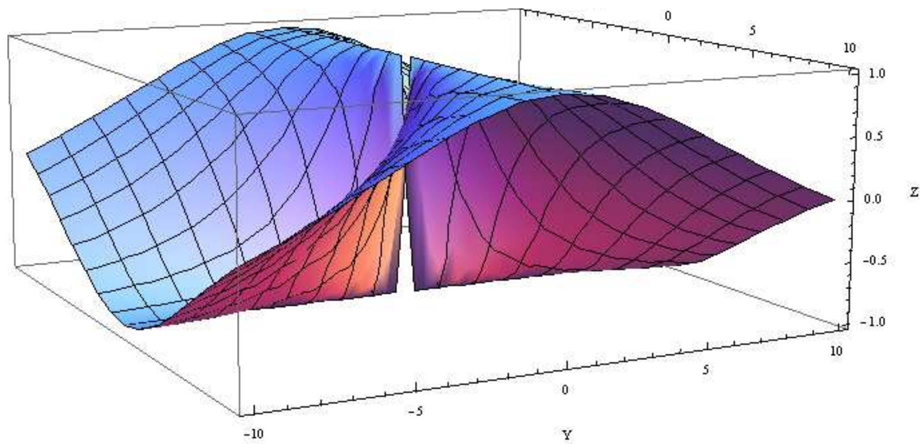
Again, this has applications to compute the value of a limit if we know beforehand that it exists. Also, if for some function $f(x, y)$ we can prove that

$$\lim_{x\to a} \lim_{y\to b} f(x, y) \neq \lim_{y\to b} \lim_{x\to a} f(x, y)$$

then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist. But, the above relations cannot be used to prove that $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.

Example 5.14. Consider the function,

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$



Note that

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

but,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

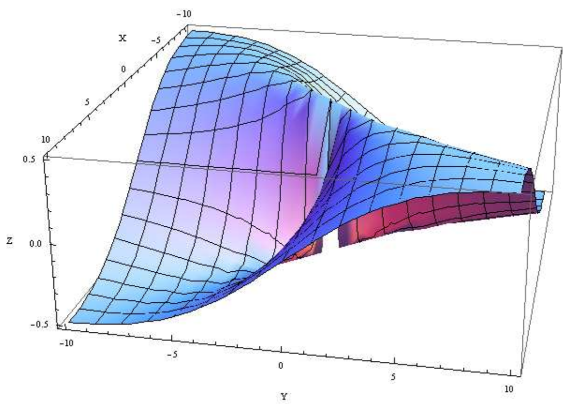
Hence, the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

does not exist.

Example 5.15. Consider the function,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$



Note that the iterated limits

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{0}{x^2} = 0$$
$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{0}{y^2} = 0$$

coincide. But, if we consider the curve, $\sigma(t) = (t,t)$ and compute

$$\lim_{t\to 0} f(\sigma(t)) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \frac{1}{2}$$

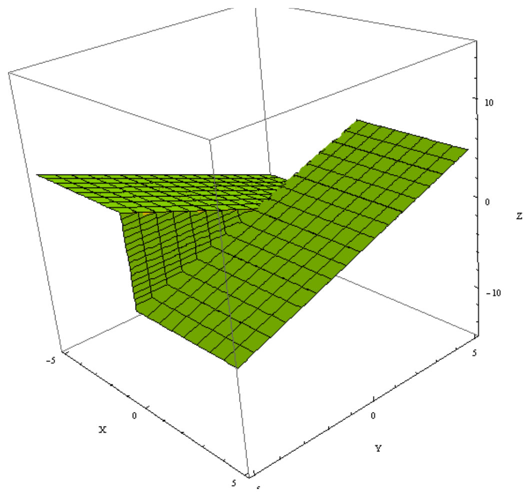
does not coincide with the value of the iterated limits. Hence, the limit

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$$

does not exist.

Example 5.16. Let

$$f(x,y) = \begin{cases} y & \text{if } x > 0 \\ -y & \text{if } x \le 0 \end{cases}$$



We show first that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. To do this, consider any $\varepsilon > 0$ and take $\delta = \varepsilon$. Now, if $0 < ||(x,y)|| = \sqrt{x^2+y^2} < \delta$ then,

$$|f(x,y) - 0| = |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta = \varepsilon$$

Hence,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

But, we remark that $\lim_{x\to 0} f(x,y)$ does not exist for $y \neq 0$. This so, because if $y \neq 0$ then the limits

$$\lim_{x\to 0^+} f(x,y) = y$$

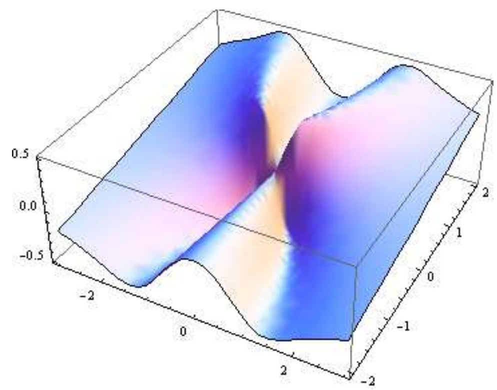
$$\lim_{x\to 0^-} f(x,y) = -y$$

do not coincide. So, $\lim_{x\to 0} f(x,y)$ does not exist for $y \neq 0$.

Example 5.17. Consider the function,

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

whose graph is the following



Note that

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0}\frac{0}{x^4} = 0$$

but,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0}\frac{0}{y^2} = 0$$

Moreover, if we consider the curve $\sigma(t) = (t,t)$ and compute

$$\lim_{t\to 0} f(t,t) = \lim_{t\to 0} f(t,t) = \lim_{t\to 0}\frac{t^3}{t^4+t^2} = 0$$

we see that it coincides with the value of the iterated limits.

Hence, one could wrongly conclude that the limit exists and

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2} = 0$$

But this is not true...Because, if we now consider the curve $\sigma(t) = (t,t^2)$ and compute

$$\lim_{t\to 0} f(t,t^2) = \lim_{x\to 0} f(t,t^2) = \lim_{t\to 0}\frac{t^4}{t^4+t^4} = \frac{1}{2}$$

Therefore, the limit

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$$

does not exist.

Theorem 5.18 (Algebra of limits)**.** Consider two funcions $f,g : D \subset \mathbb{R}^n \to \mathbb{R}$ and suppose

$$\lim_{x\to p} f(x) = L_1, \quad \lim_{x\to p} g(x) = L_2$$

Then,

(1) $\lim_{x\to p}(f(x)+g(x)) = L_1 + L_2$.
(2) $\lim_{x\to p}(f(x)-g(x)) = L_1 - L_2$.

(3) $\lim_{x\to p} f(x)g(x) = L_1L_2$.
(4) If $a \in \mathbb{R}$ then $\lim_{x\to p} af(x) = aL_1$.
(5) If, in addition, $L_2 \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$

The following two results will be very useful in proving that a limit exists

Proposition 5.19. Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ and suppose

(1) $g(x) \leq f(x) \leq h(x)$ for every x in some open disc centered at p.
(2) $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L$.

Then,

$$\lim_{x\to p} f(x) = L$$

Proposition 5.20. Suppose f is a function of the following type:

(1) A polynomial.
(2) A trigonometric or an exponential function.
(3) A logarithm.
(4) x^a, where $a \in \mathbb{R}$.

Let p be in the domain of f. Then

$$\lim_{x\to p} f(x) = f(p)$$

Example 5.21. Let us compute $\lim_{(x,y)\to(0,0)} f(x,y)$, where f is the function

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Consider the functions

$$g(x,y) = 0, \quad h(x,y) = \sqrt{x^2+y^2}$$

By Proposition 5.20, we have $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} h(x,y) = 0$. On the other hand,

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

So, $g(x,y) \leq |f(x,y)| \leq h(x,y)$. By proposition 5.19,

$$\lim_{(x,y)\to(0,0)} |f(x,y)| = 0$$

Finally, since, $-|f(x,y)| \leq f(x,y) \leq |f(x,y)|$, we apply again proposition 5.19 to conclude that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

6. Continuous Functions

Definition 6.1. A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at a point $p \in D$ if $\lim_{x\to p} f(x) = f(p)$. We say that f is continuous on D if its continuous at every point $p \in D$.

Remark 6.2. Note that a function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at a point $p \in D$ if and only if given $\varepsilon > 0$ there is some $\delta > 0$ such that if $x \in p$ verifies that $\|x - p\| \leq \delta$, then $\|f(x) - f(p)\| \leq \varepsilon$.

Remark 6.3. A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ can be written as

$$f(x) = (f_1(x), \dots, f_m(x))$$

We have the following.

Proposition 6.4. The function f is continuous at $p \in D$ if and only if for each $i = 1, \dots, m$, the function f_i are continuous at p.

Hence, from now on we will concentrate on functions $f : D \subset \mathbb{R}^n \to \mathbb{R}$.

7. Operations with continuous functions

Theorem 7.1. Let $D \subset \mathbb{R}^n$ and let $f, g : D \to \mathbb{R}$ be continuous at a point p in D. Then,

(1) $f + g$ is continuous at p.
(2) fg is continuous at p.
(3) if $f(p) \neq 0$, then there is some open set $U \subset \mathbb{R}^n$ such that $f(x) \neq 0$ for every $x \in U \cap D$ and
$$\frac{g}{f} : U \cap D \to \mathbb{R}$$
is continuous at p.

Theorem 7.2. Let $f : D \subset \mathbb{R}^n \to E$ (where $E \subset \mathbb{R}^m$) be continuous at $p \in D$ and let $g : E \to \mathbb{R}^k$ be continuous at $f(p)$. Then, $g \circ f : D \to \mathbb{R}^k$ is continuous at p.

Remark 7.3. The following functions are continuous,

(1) Polynomials
(2) Trigonometric and exponential functions.
(3) Logarithms, in the domain where is defined.
(4) Powers of funcions, in the domain where they are defined.

8. Continuity of functions and open/closed sets

Theorem 8.1. Let $f : \mathbb{R}^n \to \mathbb{R}$. Then, the following are equivalent.

(1) f is continuous on $\mathbb{R}^n$.
(2) For each open subset U of $\mathbb{R}$, the set $f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}$ is open.
(3) For each $a, b \in \mathbb{R}$, the set $f^{-1}(a, b) = \{x \in \mathbb{R}^n : a < f(x) \ < b\}$ is open.
(4) For each closed subset $V \subset \mathbb{R}$, the set $\{x \in \mathbb{R}^n : f(x) \in V\}$ is closed.
(5) For each $a, b \in \mathbb{R}$, the set $f\{x \in \mathbb{R}^n : a \leq f(x) \leq b\}$ is closed.

Corollary 8.2. Suppose that the functions $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are continuous. Let $-\infty \leq a_i \leq b_i \leq +\infty$, $i = 1, \dots, k$. Then,

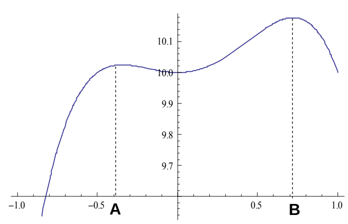
(1) The set $\{x \in \mathbb{R}^n : a_i < f_i(x) < b_i, \quad i = 1, \dots, k\}$ is open.
(2) The set $\{x \in \mathbb{R}^n : a_i \leq f_i(x) \leq b_i, \quad i = 1, \dots, k\}$ is closed.

9. Extreme points and fixed points

Definition 9.1. Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$. We say that a point $p \in D$ is a

(1) **global maximum** of f on D if $f(x) \leq f(p)$, for any other $x \in D$.
(2) **global minimum** of f on D if $f(x) \geq f(p)$, for any other $x \in D$.
(3) **local maximum** of f on D if there is some $\delta > 0$ such that $f(x) \leq f(p)$, for every $x \in D \cap B(p, \delta)$.
(4) **local minimum** of f on D if there is some $\delta > 0$ such that $f(x) \geq f(p)$, for every $x \in D \cap B(p, \delta)$.

Example 9.2. In the following picture, the point A is a local maximum but not a global one. The point B is a (local and) global maximum.



Theorem 9.3 (Weiestrass' Theorem)**.** Let $D \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$ and let $f : D \to \mathbb{R}$ be continuous. Then, there are $x_0, x_1 \in D$ such that for any $x \in D$

$$f(x_0) \leq f(x) \leq f(x_1)$$

That is, x_0 is a global minimum of f on D and x_1 is a global maximum of f on D.

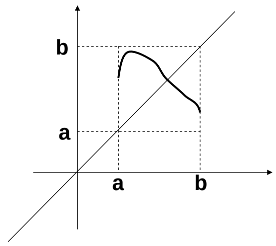
Theorem 9.4 (Brouwer's Theorem)**.** Let $D \subset \mathbb{R}^n$ be a non-empty, compact and convex subset or $\mathbb{R}^n$. Let $f : D \to D$ continuous then there is $p \in D$ such that $f(p) = p$.

Remark 9.5. If $f(p) = p$, then p is called a **fixed point** of f.

Remark 9.6. Recall that

(1) A subset of $\mathbb{R}$ is convex if and only if it is an interval.
(2) A subset of $\mathbb{R}$ is closed and convex if and only if it is a closed interval.
(3) A subset X of $\mathbb{R}$ is closed, convex and bounded if and only if $X = [a, b]$.

Example 9.7. Any continuous function $f : [a, b] \to [a, b]$ has a fixed point. Graphically,



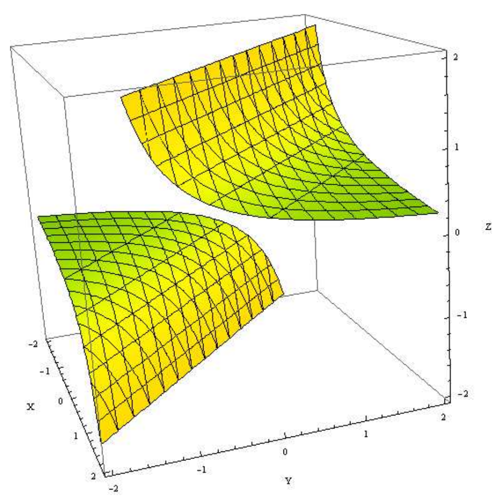
10. Applications

Example 10.1. Consider the set $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$. Since the function $f(x, y) = x^2 + y^2$ is continuous, the set A is closed. It is also bounded and hence the set A is compact.

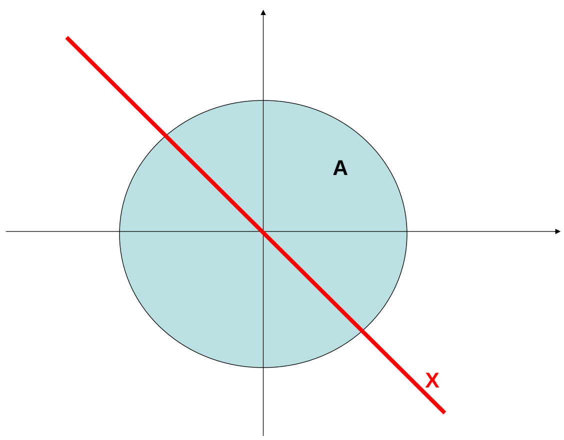
Considerer now the function

$$f(x, y) = \frac{1}{x + y}$$

Its graphic is



The function f is continuous except in the set $X = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. This set intersects A,



Taking $y = 0$, we see that

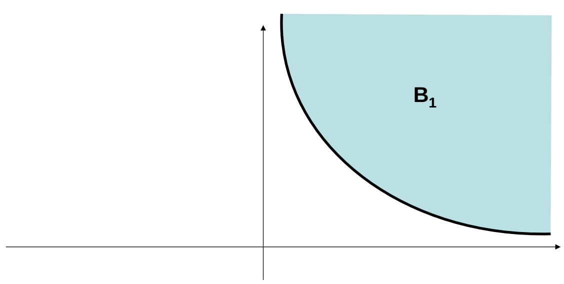
$$\lim_{\substack{x\to 0\\ x>0}} f(x,0) = +\infty \qquad \lim_{\substack{x\to 0\\ x<0}} f(x,0) = -\infty$$

and we conclude that f attains neither a maximum nor a minimum on the set A.

Example 10.2. Consider the set $B_0 = \{(x,y) \in \mathbb{R}^2 : xy \geq 1\}$. Since the function $f(x,y) = xy$ is continuous, the set B_0 is closed. Since the set B_0 is not bounded, it is not compact.

Example 10.3. How is the set $B_1 = \{(x,y) \in \mathbb{R}^2 : xy \geq 1, \quad x,y > 0\}$? Now we may not use directly the results above. But, we note that

$$B_1 = \{(x,y) \in \mathbb{R}^2 : xy \geq 1, \quad x,y > 0\} = \{(x,y) \in \mathbb{R}^2 : xy \geq 1, \quad x,y \geq 0\}$$



and since the functions $f_1(x,y) = xy$, $f_2(x,y) = x$ y $f_3(x,y) = y$ are continuous, we conclude that the set B_1 is closed. Consider again the function

$$f(x,y) = \frac{1}{x+y}$$

Does it attain a maximum or a minimum on the set B_1? Note that the function is continuous in the set B_1, we may not apply Weierstrass' Theorem because B_1 is not compact.

On the one hand, we see that $f(x, y) > 0$ in the set B_1. In addition, the points (n, n) for $n = 1, 2, \ldots$ belong to the set B_1 and

$$\lim_{n \to +\infty} f(n, n) = 0$$

Hence, given a point $p \in B_1$, we may find a natural number n large enough such that

$$f(p) > f(n, n) > 0$$

And we conclude that f does not attain a minimum in the set B_1.

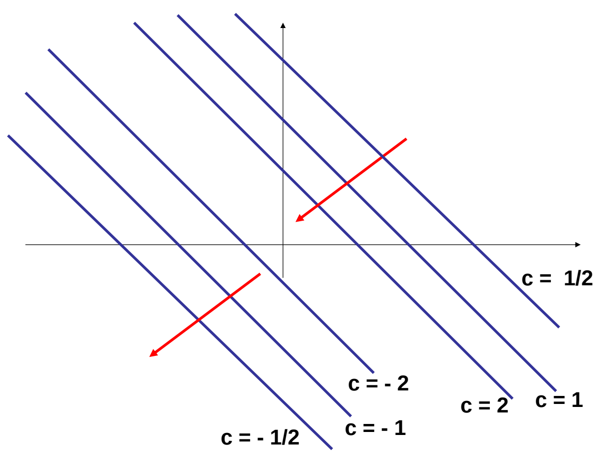
The level curves $\{(x, y) \in \mathbb{R}^2 : f(x, y) = c\}$ of the function

$$f(x, y) = \frac{1}{x + y}$$

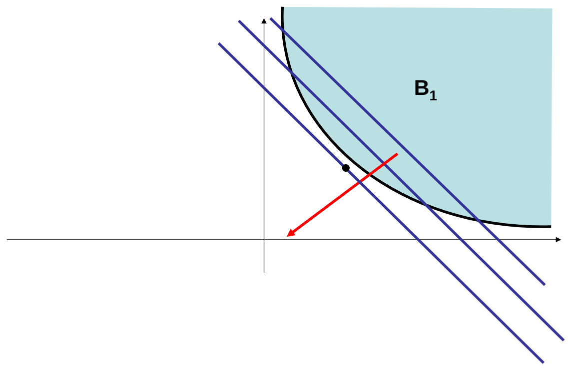
are the straight lines

$$x + y = \frac{1}{c}$$

Graphically,



The arrows point in the direction of growth of f. Graphically we see that f attains a maximum at the point of tangency with the set B_1. This is the point $(1, 1)$.



Exercise 10.4. Similarly,

$$B_2 = \{(x,y) \in \mathbb{R}^2 : xy \geq 1, \quad x,y < 0\} = \{(x,y) \in \mathbb{R}^2 : xy \geq 1, \quad x,y \leq 0\}$$

is closed, but it is not compact. Argue that the function

$$f(x,y) = \frac{1}{x+y}$$

is continuous on that set but it does not attain a maximum. On the other hand, it attains a minimum at the point $(-1,-1)$.

Exercise 10.5. The sets $B_3 = \{(x,y) \in \mathbb{R}^2 : xy > 1, \quad x,y > 0\}$ and $B_4 = \{(x,y) \in \mathbb{R}^2 : xy > 1, \quad x,y < 0\}$ are open sets. Why?