University Carlos III
Department of Economics
Mathematics II. Final Exam. May 17th 2019

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.

Problem	Points
1	
2	
3	
4	
5	
6	
Total	

(1) Given the following system of linear equations,

$$\left\{\begin{array}{rcl} 2x-y+z & = & 3 \\ x-y+z & = & 2 \\ 3x-y-az & = & b \end{array}\right.$$

where $a, b \in \mathbb{R}$ are parameters.

(a) Classify the system according to the values of a and b. **5 points**

Solution: *The matrix associated with the system is*

$$\begin{pmatrix} 2 & -1 & 1 & 3 \\ 1 & -1 & 1 & 2 \\ 3 & -1 & -a & b \end{pmatrix}$$

Exchanging rows 1 and 2 we obtain

$$(A|b) = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & -1 & 1 & 3 \\ 3 & -1 & -a & b \end{pmatrix}$$

Next, we perform the following operations

$$\textit{row 2} \mapsto \textit{row 2} - 2 \times \textit{row 1}$$

$$\textit{row 3} \mapsto \textit{row 3} + 3 \times \textit{row 1}$$

And we obtain that the original system is equivalent to another one whose augmented matrix is the following

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -a-3 & b-6 \end{pmatrix}$$

Now, we perform the operation row 3 $\mapsto$ *row 3* $- 2 \times$ *row 2 and we obtain*

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -a-1 & b-4 \end{pmatrix}$$

We see that

(i) *if* $a \neq -1$ *the system is consistent with a unique solution.*

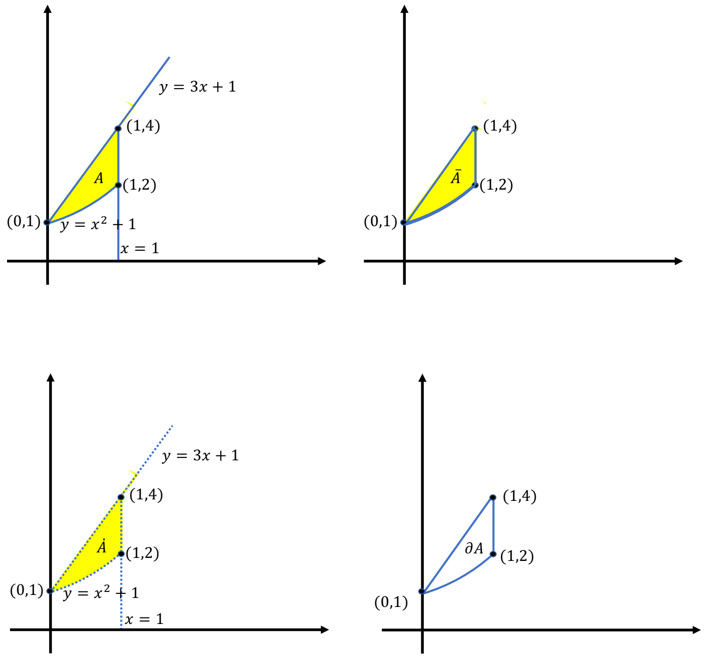
(ii) *If* $a = -1$ *the system is consistent if and only if* $b = 4$. *In the latter case, the system is underdetermined with one parameter.*

(b) Solve the above system for the values $a = -1$, $b = 4$. **3 points**

Solution: *The proposed system of linear equations is equivalente to the following one*

$$\left\{\begin{array}{lcr} x-y+z & = & 2 \\ y-z & = & -1 \end{array}\right.$$

Choosing z *as the parameter, the set of solutions is* $\{(1, z-1, z) : z \in \mathbb{R}\}$.



(2) Consider the set $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 + 1 \leq y \leq 3x + 1\}$ and the function

$$f(x, y) = \sqrt{\log(x + y)}$$

defined on A.

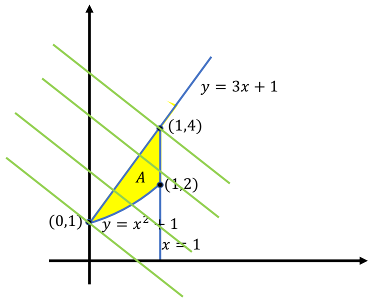
(a) Sketch the graph of the set A and justify if it is open, closed, bounded, compact or convex. **5 points**

Solution: *The set A is approximately as indicated in the picture. It is closed because $\partial A \subset A$. It is not open because $A \cap \partial A \neq \emptyset$. It is bounded. Therefore, the set A is compact. It is convex.*

(b) State Weierstrass' Theorem. Determine if it is possible to apply Weierstrass' Theorem to the function f defined on A. Using the level curves, determine (if they exist) the extreme global points of f on the set A. **5 points**

Solution: *Since, $x + y > 0$ in the set A, the function $f(x, y) = \sqrt{\log(x + y)}$ is continuous and Weierstrass' Theorem may be applied. The function f attains a maximum and a minimum on A The level curves are of the form $x + y = c$. In the picture we represent the level curves in green color.*

Graphically, we see that the maximum value is attained at the point $(1, 4)$ and the minimum value is attained at the point $(0, 1)$.



(3) Consider the function $f(x,y) = 2x + y - \ln x - \ln y$.

(a) Determine its domain and the regions of $\mathbb{R}^2$ where the function is concave or convex. **5 points**

Solution: *The domain of the function is* $Dom(f) = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ *The gradient of the function is*

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2 - \frac{1}{x} & 1 - \frac{1}{y} \end{pmatrix}$$

We obtain now the Hessian matrix

$$Hf(x,y) \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{1}{x^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

The associated quadratic form is positive definite. This follows from the principal dominant minors

$$D_1 = \frac{1}{x^2}, \quad D_2 = \frac{1}{x^2 y^2}$$

Hence, $D_1 > 0$ *and* $D_2 > 0$ *at every point of* $\mathbb{R}^2$*. Therefore,* Hf *is positive definite in the domain of* f*. It follows that* f *is strictly convex in its domain.*

(b) Study the existence of global extreme points for the function f in its domain. **5 points**

Solution: *The critical points are solutions of*

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 2 - \frac{1}{x} & 1 - \frac{1}{y} \end{pmatrix} = (0,0)$$

We obtain a unique critical point $\left(\frac{1}{2}, 1\right)$*. Since, the function* f *is strictly convex, the unique critical point is the (only) global minimum of* f *in its domain.*

(4) Consider the set of equations

$$\begin{aligned} x^2y + ze^y &= -1 \\ x - y + z &= 0 \end{aligned}$$

(a) Prove that the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (1, 0, -1)$. **3 points**

Solution: *The functions $f_1(x, y, z) = x^2y + ze^y - 1$ and $f_2(x, y, z) = x - y + z$ are of class C^∞. We compute*

$$\begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix}_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} x^2 + ze^y & e^y \\ -1 & 1 \end{vmatrix}_{(x,y,z)=(1,0,-1)} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$$

By the Implicit function theorem the above system of equations determines implicitly two differentiable functions $y(x)$ and $z(x)$ in a neighborhood of the point $(x, y, z) = (1, 0, -1)$.

(b) Compute

$$y'(1), z'(1)$$

and the first order Taylor polynomial of $y(x)$ and $z(x)$ at the point $x_0 = 1$. **5 points**

Solution: *Differentiating implicitly with respect to x,*

$$\begin{aligned} 2xy + x^2y' + e^y zy' + e^y z' &= 0 \\ 1 - y' + z' &= 0 \end{aligned}$$

We plug in the values $(x, y, z) = (1, 0, -1)$ to obtain the following

$$\begin{aligned} z'(1) &= 0 \\ 1 - y'(1) + z'(1) &= 0 \end{aligned}$$

So,

$$z'(1) = 0, \quad y'(1) = 1$$

Thus, Taylor's polynomial of order 1 of the function $y(x)$ at the point $x_0 = 1$ is

$$P_1(x) = y(1) + y'(1)(x - 1) = x - 1$$

and Taylor's polynomial of order 1 of the function $z(x)$ at the point $x_0 = 1$ is

$$Q_1(x) = z(1) + z'(1)(x - 1) = -1$$

(5) Consider the function $f(x, y) = 3axy - x^3 - y^3$, where $a \neq 0$ is a parameter.

(a) Determine the critical points of f in the set $\mathbb{R}^2$. **5 points**

(b) Find and classify, according to the values of the parameter a, the critical points of f. **5 points**

(c) Determine the value of the parameter a for which there is a local maximum where the function attains the value 8 and the value of the parameter a for which there is a local minimum where the function attains the value -1. **5 points**

Solution:

(a) *We compute the gradient of* f

$$\nabla f(x, y) = (3ay - 3x^2, 3ax - 3y^2)$$

The critical points are the solutions of the system of equations

$$3ay - 3x^2 = 0, 3ax - 3y^2 = 0$$

The solutions are

$$(0, 0) \quad \textit{and} \quad (a, a)$$

(b) *The Hessian matrix is*

$$Hf(x, y) = \begin{pmatrix} -6x & 3a \\ 3a & -6y \end{pmatrix}$$

At the point $(0, 0)$ *we obtain*

$$Hf(0, 0) = \begin{pmatrix} 0 & 3a \\ 3a & 0 \end{pmatrix}$$

So, $D_2 = -9a^2 < 0$. *The associated quadratic form is indefinite. The point* $(0, 0)$ *is a saddle point.*
At the point (a, a) *we obtain*

$$Hf(a, a) = \begin{pmatrix} -6a & 3a \\ 3a & -6a \end{pmatrix}$$

So, $D_1 = -6a$, $D_2 = 27a^2 > 0$.

(i) *If* $a < 0$, *the associated quadratic form is positive definite. The point* (a, a) *corresponds to a local minimum.*

(ii) *If* $a > 0$, *the associated quadratic form is negative definite. The point* (a, a) *corresponds to a local maximum.*

Since,

$$\lim_{x \to \infty} f(x, 0) = -\infty, \qquad \lim_{x \to -\infty} f(x, 0) = \infty$$

the above points do not correspond to global extreme points.

(c) *The value of the function at the point* a *is*

$$f(a, a) = a^3$$

(i) *If* $a = 2$, *the point* $(2, 2)$ *corresponds to a local maximum and* $f(2, 2) = 8$.

(ii) *If* $a = -1$, *the point* $(-1, -1)$ *corresponds to a local minimum and* $f(-1, -1) = -1$.

(6) Consider the functions $f(x,y) = xy$ and $g(x,y) = x^2 + y^2 - 8$. Let $S = \{(x,y) : g(x,y) = 0\}$.

(a) Explain why $f(x,y)$ must have a global maximum on the set S. **2 points**

(b) Using the Lagrangian method, find the global maxima of $f(x,y)$ on the set S. **7 points**

Solution:

(a) *The set is bounded and it consists of all its boundary points. Therefore it is compact. The function* f *is a polynomial function, therefore continuous. Since* A *is compact, the result follows from the Extreme Value Theorem (Weierstrass).*

(b) *The function* f *is continuously differentiable, and there are no feasible irregular points since* $\nabla g(x,y) = (2x, 2y)$ *and* $(0,0) \notin S$. *Now we can be sure that all candidates for the global maximum must satisfy the first-order necessary conditions. The Lagrangian is*

$$\mathcal{L}(x,y) = xy - \lambda(x^2 + y^2 - 8)$$

The first-order necessary conditions are:

$$\mathcal{L}_x(x,y) = \quad y - 2\lambda x = 0 \tag{1}$$

$$\mathcal{L}_y(x,y) = \quad x - 2\lambda y = 0 \tag{2}$$

$$x^2 + y^2 - 8 = 0 \tag{3}$$

Subtracting the two first-order conditions, we get $(y - x)(1 - 2\lambda) = 0$. *If* $x = y$, *then condition (3) says* $2x^2 = 8$, *therefore the candidates are* $(2,2)$ *or* $(-2,-2)$. *If* $x \neq y$, *then* $\lambda = -1/2$. *This means that* $y = -x$ *from (1). As a result, the remaining candidates are* $(-2,2)$ *and* $(2,-2)$. *Given the absence of irregular points or corner solutions, we only have to consider these four candidates. Since* $f(2,-2) = f(-2,2) = -4$ *and* $f(2,2) = f(-2,-2) = 4$, *the global maximum is* $f(2,2) = f(-2,-2) = 4$ *which occurs at two different points* $(2,2), (-2,-2)$.