$\boxed{1}$

Consider the following system of linear equations depending on the parameter $a \in \mathbb{R}$,

$$\left\{\begin{array}{rrrcl} x & +y & -z & = & 1 \\ x & -y & +z & = & 7 \\ -x & +y & +z & = & 3 \\ 2x & +ay & -4z & = & a \end{array}\right.$$

(a) (6 points) Classify the system according to the values of the parameter a.

(b) (4 points) Solve the above system for those values of a for which the system has a unique solution.

Solution:

(a) We use the Gauss method to transform the extended matrix of the system

$$A^* = \left(\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 7 \\ -1 & 1 & 1 & 3 \\ 2 & a & -4 & a \end{array}\right),$$

into an equivalent echelon matrix.

$$\left(\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 7 \\ -1 & 1 & 1 & 3 \\ 2 & a & -4 & a \end{array}\right) \overset{\substack{f_2-f_1 \\ f_3+f_1 \\ f_4-2f_1}}{\sim} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 6 \\ 0 & 2 & 0 & 4 \\ 0 & a-2 & -2 & a-2 \end{array}\right)$$

$$\overset{\substack{f_3+f_2 \\ 2f_4+(a-2)f_2}}{\sim} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 2(a-4) & 8(a-2) \end{array}\right) \overset{f_4-(a-4)f_3}{\sim} \left(\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & 6 \\ 0 & 0 & 2 & 10 \\ 0 & 0 & 0 & -2a+24 \end{array}\right).$$

Thus, we have:

- If $a = 12$, then $\text{rank}(A) = \text{rank}(A^*) = 3$, and so the system has a unique solution.
- Otherwise, if $a \neq 12$, then $\text{rank}(A) = 3 \neq 4 = \text{rank}(A^*)$, and so the system has no solutions.

(b) By considering $a = 12$, the equivalent echelon system is

$$\left\{\begin{array}{rrrcr} x & +y & -z & = & 1 \\ & -2y & +2z & = & 6 \\ & & 2z & = & 10 \end{array}\right.$$

whose solution is $(x = 4, y = 2, z = 5)$.

$\boxed{2}$

Let consider the set $A = \{(x, y) \in \mathbb{R}^2 \,:\, y \geq x^2 - 2x,\ y \leq 2x - x^2\}$ and the function $f(x, y) = x + y$.

(a) (4 points) Draw the set A. Prove that the function f attains global maximum and global minimum values in the set A.

(b) (6 points) Draw the level curves of f. Using this information, find the points where the global maximum and global minimum values of f in A are attained, and the value of f in those points.

Solution:

(a) The set A is the darken region represented in the figure below.

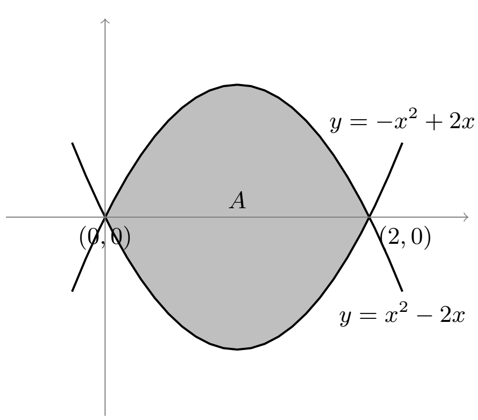
The set A is closed, as it contains all of its boundary points. It is also bounded, as it can be enclosed in a finite ball, for instance, the closed ball centered in $(0, 0)$ and radius 2. Consequently, A is compact. The function f is continuous in $\mathbb{R}^2$ because it is linear, and so f is continuous in A. Hence, by the Weierstrass Theorem, the function f attains global extrema (maximum and minimum) in A.

(b) The level curves of f are of the form $x + y = k$, with $k \in \mathbb{R}$. They are straight lines with slope -1 crossing the point $(0, k)$. Hence, according to the graph, the global minimum (maximum, respectively) of f in A is attained at the point in which the level curve is tangent to the curve $y = x^2 - 2x$ ($y = -x^2 + 2x$, respectively).

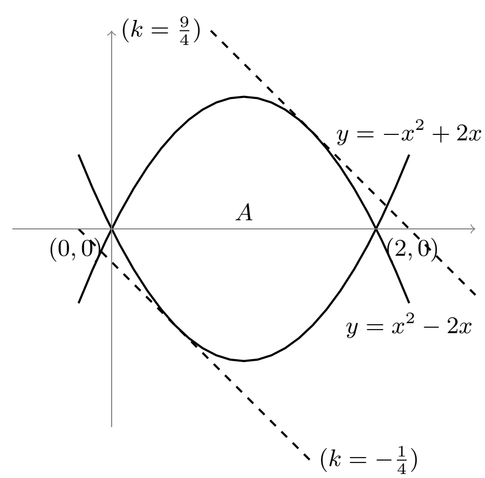
The derivative of $y = x^2 - 2x$ is $y' = 2x - 2$, and the derivate of the level curves are -1. The solution of $2x - 2 = -1$ is $x = \frac{1}{2}$. The corresponding y component is then $y = (\frac{1}{2})^2 - 2\frac{1}{2} = -\frac{3}{4}$. Hence, f attains a global minimum in A at the point $(\frac{1}{2}, -\frac{3}{4})$, and the minimum value of f in A is then $-\frac{1}{4}$.

The derivative of $y = -x^2 + 2x$ is $y' = -2x + 2$, and the derivate of the level curves are -1. The solution of $-2x + 2 = -1$ is $x = \frac{3}{2}$. The corresponding y component is then $y = -(\frac{3}{2})^2 + 2\frac{3}{2} = \frac{3}{4}$. Hence, f attains a global maximum in A at the point $(\frac{3}{2}, \frac{3}{4})$, and the maximum value of f in A is then $\frac{9}{4}$.

Set



Level curves



$\boxed{3}$

Let $a \in \mathbb{R}$ and consider the function $f(x, y, z) = ax^2 + ay^2 - 6x - z + (a-1)z^4$

(a) (5 points) Study the concavity and convexity of f depending on the values of the parameter a.

(b) (5 points) For $a = 3$, find the local and global extreme points of f, in case they exist.

Solution:

(a) The function f is of class $\mathcal{C}^2$ in $\mathbb{R}^2$ as it is a polynomial. The gradient of f is

$$\nabla f(x, y, z) = (2ax - 6, 2ay, -1 + 4(a-1)z^3).$$

The Hessian matrix of f is

$$\begin{pmatrix} 2a & 0 & 0 \\ 0 & 2a & 0 \\ 0 & 0 & 12(a-1)z^2 \end{pmatrix}$$

which is a diagonal matrix, and so it is easier to determine its definiteness. The diagonal elements are $2a$ and $12(a-1)z^2$.

- If $a \geq 1$, then $2a > 0$ and $12(a-1)z^2 \geq 0$ for any z. Hence, the Hessian matrix is positive semidefinite for any z and f is then convex.
- If $0 < a < 1$, then $2a > 0$ and $12(a-1)z^2 < 0$ for all $z \neq 0$. Hence, the Hessian matrix is indefinite and f is neither convex nor concave.
- If $a \leq 0$, then $2a \leq 0$ and $12(a-1)z^2 \leq 0$ for all z. Hence, the Hessian matrix is negative semidefinite for any z and f is then concave.

(b) The critical points are the solutions of the system

$$\left.\begin{array}{rcl} 6x - 6 & = & 0 \\ 6y & = & 0 \\ -1 + 8z^3 & = & 0 \end{array}\right\},$$

whose solution is $p = (1, 0, \frac{1}{2})$. Since we are assuming $a = 3 \geq 1$, then f is convex and so the point p is both local and global minimum of f.

$\boxed{4}$

Let's optimize the function $f(x,y) = x^2 + y^2 + 3xy$ in the set $A = \{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}$.

(a) (4 points) Find the Lagrange equations of the problem.

(b) (6 points) Determine the extreme points of f in A and classify them into maxima or minima.

Solution:

(a) The set A is compact and the function f is continuous and differentiable as it is a polynomial. Hence, f attains global extrema in A. The Lagrangian function is

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + 3xy + \lambda(1 - x^2 - y^2)$$

and the Lagrange equations are

$$\left\{\begin{array}{rcl} 2x + 3y - 2\lambda x & = & 0 \\ 2y + 3x - 2\lambda y & = & 0 \\ x^2 + y^2 & = & 1 \end{array}\right.$$

(b) We must solve the Lagrange equations. As $(0,0)$ does not satisfy the last equation, the homogeneous linear system

$$\left\{\begin{array}{rcl} 2(1-\lambda)x + 3y & = & 0 \\ 3x + 2(1-\lambda)y & = & 0 \end{array}\right. ,$$

with determinant $4(1-\lambda)^2 - 9$, has a solution different from $(0,0)$ if and only if $4(1-\lambda)^2 - 9 = 0$, that is, $\lambda = -\frac{1}{2}$ or $\lambda = \frac{5}{2}$. Hence, $y = -x$ for $\lambda = -\frac{1}{2}$, and $y = x$ for $\lambda = \frac{5}{2}$, respectivamente. From the last equation we get the candidates

$$P_1 = (1/\sqrt{2}, -1/\sqrt{2}),\ P_2 = (-1/\sqrt{2}, 1/\sqrt{2}),\ P_3 = (1/\sqrt{2}, 1/\sqrt{2}),\ P_4 = (-1/\sqrt{2}, -1/\sqrt{2}).$$

The value of f is $-\frac{1}{2}$ in P_1 and P_2, and $\frac{5}{2}$ in P_3 and P_4, so P_1, P_2 are global minima and P_3, P_4 are global maxima.

$\boxed{5}$

Consider the following system of equations

$$\left.\begin{array}{rcl} xy + 2yz + zt^2 & = & 5 \\ x^2 z + y^2 t & = & 4 \end{array}\right\}.$$

(a) (4 points) Determine whether the Implicit Function Theorem can be applied to this system to assert that the variables z and t are differentiable functions of x and y in a neighbourhood of the point $P = (x_0, y_0.z_0, t_0) = (2, 1, \frac{1}{2}, 2)$.

(b) (6 points) Compute the values of $\dfrac{\partial z}{\partial x}(2,1)$ and $\dfrac{\partial t}{\partial x}(2,1)$.

Solution:

(a) Let us check the hypothesis of the Implicit Function Theorem. We call $f_1(x,y,z,t) := xy + 2yz + zt^2 - 5$ and $f_2(x,y,z,t) := x^2 z + y^2 t - 4$. One has,

- $f_1, f_2 \in \mathcal{C}^1$ since they are both polynomials
- $f_1(2,1,\frac{1}{2},2) = f_2(2,1,\frac{1}{2},2) = 0$, as

$$\left.\begin{array}{rcl} 2\cdot 1 + 2\cdot 1 \cdot \frac{1}{2} + \frac{1}{2}\cdot 2^2 & = & 5 \\ 2^2 \cdot \frac{1}{2} + 1^2 \cdot 2 & = & 4 \end{array}\right\}$$

- Last condition is also satisfied as

$$\frac{\partial(f_1,f_2)}{\partial(z,t)}(2,1,\frac{1}{2},2) = \left|\begin{matrix} \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial t} \end{matrix}\right|_{(2,1,\frac{1}{2},2)} = \left|\begin{matrix} 2y+t^2 & 2zt \\ x^2 & y^2 \end{matrix}\right|_{(2,1,\frac{1}{2},2)} = \left|\begin{matrix} 6 & 2 \\ 4 & 1 \end{matrix}\right| = -2 \neq 0$$

Hence, the above system defines the variables z and t as differentiable functions of x and y in a neighbourhood of the point $(2,1,\frac{1}{2},2)$, and we know that $z(2,1) = \frac{1}{2}$ and $t(2,1) = 2$.

(b) We first derive the original system respect to x.

$$\left.\begin{array}{rcl} y + 2y\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}t^2 + z2t\frac{\partial t}{\partial x} & = & 0 \\ 2xz + x^2\frac{\partial z}{\partial x} + y^2\frac{\partial t}{\partial x} & = & 0 \end{array}\right\}$$

Now, we substitute at the point $(2,1,\frac{1}{2},2)$ and we get the linear system

$$\left.\begin{array}{rcl} 1 + 6\frac{\partial z}{\partial x}(2,1) + 2\frac{\partial t}{\partial x}(2,1) & = & 0 \\ 2 + 4\frac{\partial z}{\partial x}(2,1) + \frac{\partial t}{\partial x}(2,1) & = & 0 \end{array}\right\}$$

whose solution is $\frac{\partial z}{\partial x}(2,1) = -\frac{3}{2}$ and $\frac{\partial t}{\partial x}(2,1) = 4$.

$\boxed{6}$

Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2y - y^3}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) (4 points) Study the continuity of f.

(b) (6 points) Find the partial derivatives

$$\frac{\partial f}{\partial x}(0,0), \quad \frac{\partial f}{\partial y}(0,0)$$

and the directional derivative of f at $(1,1)$ along the direction $(4/5, 3/5)$.

Solution:

(a) Note that $\operatorname{dom} f = \mathbb{R}^2$. It is clear that f is continuous in the open set $S := \mathbb{R}^2 - \{(0,0)\}$ since it is well defined and it is a quotient of polynomials.

Let us study whether f is continuous at $(0,0)$. The iterated limits are both equal to 0. If we approach $(0,0)$ by any curve, the limit is also 0. We will prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - y^3}{x^2+y^2} = 0. \tag{1}$$

Let $\varepsilon > 0$. Then

$$\left| \frac{x^2y - y^3}{x^2+y^2} - 0 \right| = \left| \frac{x^2y - y^3}{x^2+y^2} \right| = |y| \cdot \left| \frac{x^2 - y^2}{x^2+y^2} \right| \le |y| \le \sqrt{x^2+y^2} < \delta = \varepsilon$$

for all $(x,y) \in B((0,0),\delta)$, since $|x^2 - y^2| \le x^2 + y^2$. Letting $\delta = \varepsilon$, (1) holds.

Therefore, f is continuous in $\mathbb{R}^2$.

(b) We first compute the partial derivatives

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(0,t)}{t} = \lim_{t\to 0} \frac{-t}{t} = -1$$

Now, let us compute the directional derivative. For $(x,y) \neq (0,0)$ we have

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) = \left(\frac{4xy^3}{(x^2+y^2)^2}, \frac{x^4 - 4x^2y^2 - y^4}{(x^2+y^2)^2} \right)$$

The partial derivatives are continuous functions in the open set S, and so f is differentiable in S and in particular, in the point $(1,1)$. Consequently, in order to compute the directional derivative we may use the formula $D_{(v_1,v_2)} f(x_0,y_0) = (v_1,v_2) \cdot \nabla f(x_0,y_0)$. Thus,

$$D_{(\frac{4}{5},\frac{3}{5})} f(1,1) = (\tfrac{4}{5},\tfrac{3}{5}) \cdot \nabla f(1,1) = (\tfrac{4}{5},\tfrac{3}{5}) \cdot (1,-1) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}.$$