University Carlos III
Department of Economics
Mathematics II. Final Exam. June 2012

Last Name:		**Name:**
ID number:	**Degree:**	**Group:**

IMPORTANT

- **DURATION OF THE EXAM: 2h**
- Calculators are **NOT** allowed.
- **Scrap paper:** You may use the last two pages of this exam and the space behind this page.
- **Do NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor.
- Read the exam carefully. Each part of the exam counts 1 point. Please, check that there are 10 pages in this booklet

Problem	Points
1	
2	
3	
4	
5	
Total	

(1) *Consider the following system of linear equations*

$$\begin{cases} x - by - 2z &= 1 \\ x - az &= b \\ x + (2-b)y &= 1 \end{cases}$$

where $a, b \in \mathbb{R}$.

(a) *Classify the system according to the values of* a *and* b.

(b) *Solve the above system for the values of* a *and* b *for which the system has infinitely many solutions.*

Solution:

(a) The augmented matrix of the system is

$$(A|B) = \begin{pmatrix} 1 & -b & -2 & 1 \\ 1 & 0 & -a & b \\ 1 & 2-b & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -b & -2 & 1 \\ 0 & b & 2-a & b-1 \\ 1 & 2-b & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b & -2 & 1 \\ 0 & b & 2-a & b-1 \\ 0 & 2 & 2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -b & -2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & b & 2-a & b-1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & b & 2-a & b-1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -b & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -a-b+2 & b-1 \end{pmatrix}$$

Looking at the possible ranks of A and comparing with those of $(A|B)$, we can say that if $a+b \neq 2$, then $\text{rank}(A) = \text{rank}(A|B) = 3$. In those cases the system is consistent with unique solution.
If $a+b=2$ and $b \neq 1$ the system is inconsistent, as $\text{rank}(A) = 2 < rg(A|B) = 3$.
If $a+b=2$ and $b=1$ the system is undetermined and we will need one parameter to give the solution because $\text{rank}(A) = 2 = rg(A|B)$.

(b) The system is undetermined if $a = 1, b = 1$. For those values, the above system is equivalent to the following one

$$\begin{cases} x - y - 2z &= 1 \\ y + z &= 0 \end{cases}$$

whose solution is $y = -z$, $x = 1 + z$, $z \in \mathbb{R}$.

(2) *Consider the function*

$$f(x,y)=\begin{cases}\frac{x^2\sqrt{|y|}}{x^2+y^2} & \text{si } (x,y)\neq(0,0),\\ 0 & \text{si } (x,y)=(0,0).\end{cases}$$

(a) *Determine whether the function f is continuous at the point $(0,0)$.*

(b) *Compute (if they exist) the partial derivatives of f at the point $(0,0)$. Compute (if it exists) the derivative of f at the point $(0,0)$ according to the vector $v=(1,4)$. ¿Is the function f differentiable at the point $(0,0)$?*

Solution:

(a) Let $(x,y)\neq(0,0)$, we have

$$|f(x,y)|=\frac{x^2\sqrt{|y|}}{x^2+y^2}=\frac{x^2}{x^2+y^2}\sqrt{|y|}\leq\sqrt{|y|}$$

so when $(x,y)\neq(0,0)$,

$$0\leq|f(x,y)|\leq\sqrt{|y|}$$

The function $h(x,y)=\sqrt{|y|}$ is continuous and $\lim_{(x,y)\to(0,0)}h(x,y)=0$. Using Sandwich's theorem we can state that $\lim_{(x,y)\to(0,0)}f(x,y)=0$, thus the function f is continuous.

(b) The partial derivatives of f at the point $(0,0)$ are

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{t\to0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to0}\frac{0}{t}=0\\ \frac{\partial f}{\partial x}(0,0) &= \lim_{t\to0}\frac{f(0,t)-f(0,0)}{t}=\lim_{t\to0}\frac{0}{t}=0\end{aligned}$$

the derivative of f at the point $(0,0)$ in the direction of the vector $v=(1,4)$ is

$$D_vf(0,0)=\lim_{t\to0}\frac{f(t,4t)-f(0,0)}{t}=\lim_{t\to0}\frac{2\sqrt{|t|}}{17t}$$

hence it doesn't exist. Therefore, we can conclude that the function is not differentiable at $(0,0)$.

(3) *Consider the function* $f(x, y) = ax^2 + (a + b)y^2 + 2axy + 2$, *with* $a, b \in \mathbb{R}$

(a) *Study the concavity and the convexity of the function* f *according to the values of* a *and* b.

(b) *For the values* $a = 1$, $b = 0$, *does the function* f *attain a maximum and/or minimum value in* $\mathbb{R}^2$*? At what points? (Justify the answer).*

Solution:

(a) the Hessian matrix of f is

$$\mathrm{H}\, f(x, y) = \begin{pmatrix} 2a & 2a \\ 2a & 2(a + b) \end{pmatrix}$$

The principal minors are $D_1 = 2a$, $D_2 = 4ab$. The function f is convex if $a > 0$, $b \geq 0$ and it is concave if $a < 0$, $b \leq 0$. If $a = 0$ the Hessian matrix is

$$\mathrm{H}\, f(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & 2b \end{pmatrix}$$

Therefore, it is positive semidefinite if $b \geq 0$, so it is convex and it is negative semidefinite if $b \leq 0$ and convave.

(b) Substituting $a = 1$, $b = 0$, in the function, it is convex on $\mathbb{R}^2$. The critical points are given by the equations

$$2x + 2y = 0, \quad 2y + 2x = 0$$

whose solution is $y = -x$. Because f is convex and differentiable, the critical points of f are the global minima of f on $\mathbb{R}^2$. Therefore, f attains it's minimum value at the points of the form $(x, -x)$, $x \in \mathbb{R}$.

(4) *Consider the function*

$$f(x, y) = xy^2$$

and the set $A = \{(x, y) \in \mathbb{R}^2 : x + y \leq 100, 2x + y \leq 120\}$.

(a) *Compute the Kuhn-Tucker equations that determine the extreme points of* f *in* A.

(b) *Compute the solutions of the above Kuhn-Tucker equations.*

Solution:

(a) The Lagrangian associated with the problem is

$$L = xy^2 + \lambda(100 - x - y) + \mu(120 - 2x - y)$$

and the Kuhn-Tucker equations are

$$\begin{aligned} y^2 - \lambda - 2\mu &= 0 \\ 2xy - \lambda - \mu &= 0 \\ \lambda(100 - x - y) &= 0 \\ \mu(120 - 2x - y) &= 0 \\ x + y &\leq 100 \\ 2x + y &\leq 120 \end{aligned}$$

with λ, $\mu \geq 0$ for the maxima and λ, $\mu \leq 0$ for the minima.

(b) **Case 1:** $\lambda \neq 0$. Then, $x + y = 100$, or, $y = 100 - x$. Thus, $120 - 2x - y = 20 - x$. It implies that $\mu(20 - x) = 0$.

Suppose that $\mu \neq 0$. Then, $x = 20$, $y = 80$. Subtracting the first two equations, we obtain $\mu = y^2 - 2xy = 3200$. From this we can deduce that $\lambda = y^2 - 2\mu = 0$, which is contrary to $\lambda \neq 0$. Suppose that $\mu = 0$. then, $y^2 = 2xy$. From the first equation we deduce that $y^2 = \lambda \neq 0$. Then, $y = 2x$, together with $x + y = 100$, implies $x = 100/3$, $y = 200/3$. But, those values don't satisfy $2x + y \leq 120$.

Case 2: $\lambda = 0$. The Kuhn-Tucker equations are

$$\begin{aligned} y^2 - 2\mu &= 0 \\ 2xy - \mu &= 0 \\ \mu(120 - 2x - y) &= 0 \\ x + y &\leq 100 \\ 2x + y &\leq 120 \end{aligned}$$

with λ, $\mu \geq 0$ for the maxima and λ, $\mu \leq 0$ for the minima.

If $\mu = 0$, then $y = 0$ we get the solution $y = \mu = \lambda = 0$, $x \leq 60$ and those are critical points for both local maxima and local minima. If $\mu \neq 0$, then $y \neq 0$ y $2x + y = 120$. From the first two equations we obtain $y^2 = 4xy$. Therefore, $y = 4x$, together with $2x + y = 120$, implies the solution $x = 20$, $y = 80$, $\mu = 3200$, $\lambda = 0$.

(5) *Consider the function*

$$f(x, y) = x^2 + y^2$$

and the set $A = \{(x, y) \in \mathbb{R}^2 : x - 2y + 6 = 0\}$.

(a) *Compute the Lagrange equations that determine the extreme points of* f *in* A *and find their solutions.*

(b) *Using the second order conditions, characterize the above solutions of the Lagrange equations as corresponding to maximum or minimum values. Can you justify if any of those points corresponds to a global maximum or minimum?* (Justify the answer)

Solution:

(a) The Lagrangian is

$$L = x^2 + y^2 - \lambda(x - 2y + 6)$$

and we obtain the Lagrange equations,

$$\begin{aligned} 2x - \lambda &= 0 \\ 2y + 2\lambda &= 0 \\ x - 2y + 6 &= 0 \end{aligned}$$

or equivalently,

$$\begin{aligned} \lambda &= 2x \\ y + \lambda &= 0 \\ x - 2y + 6 &= 0 \end{aligned}$$

We need to solve that system of equations. From the first equation we obtain

$$x = \frac{\lambda}{2}$$

and from the second equation

$$y = -\lambda$$

Substituting those in the third one we get

$$\frac{\lambda}{2} + 2\lambda + 6 = 0$$

whose solution is

$$\lambda = \frac{-12}{5}$$

Hence,

$$x = \frac{-6}{5}, y = \frac{12}{5}, \lambda = \frac{-12}{5}$$

(b) The Hessian matrix associated with L is

$$\mathbf{H} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

And because H L is positive definite, the given critical point is a strict local minimum. Futhermore, because the function is convex (H f is positive definite), we conclude that it is a global minimum.