University Carlos III of Madrid

Department of Economics
Mathematics II. Final Exam. June 2005.

IMPORTANT:

- **DURATION OF THE EXAM: 2h. 30min.**
- Calculators are **NOT** allowed.
- **Hand in this booklet.** Do not hand in scratch paper. Only the answers written on this booklet will be graded.
- You must show a valid ID to the professor.
- Each part of the exam counts $0'5$ points.

Last Name: **Name:**

DNI: **Group:**

(1) Consider the following system of linear equations

$$\begin{aligned} x+y+z &= 1 \\ ax+ay+az &= b \\ x+ay+az &= 3 \end{aligned}$$

where a and b are parameters.

(a) Study the system according to the values of the parameters a and b and determine whether the system is determinate compatible, indeterminate compatible or incompatible.

(b) Solve the previous system for the values of a and b that make the system indeterminate compatible.

(2) Given the matrix

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 0 & 0 & a \end{pmatrix}$$

(a) Compute the characteristic polynomial and the eigenvalues.

(b) For $a=-1$, check that the matrix A is diagonalizable and find the matrix change of basis.

(c) Study for which values $a \neq -1$ the matrix A is diagonalizable.

(3) Given the quadratic form $Q(x,y,z) = x^2 - 2axy + y^2 + z^2 - 2axz$,

(a) Determine for what values of a, the quadratic form Q is positive definite.

(b) Determine for what values of a, the quadratic form Q is indefinite.

(4) Consider $A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 2, y \geq x^2\}$.

(a) Draw the set A, its boundary and its interior. Is A closed, open, convex, bounded or compact?

(b) State Weierstrass Theorem. Can this theorem be applied to the function

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

in the set A?

(c) Determine if the function $f(x,y)$ above attains a maximum or a minimum in the set A.
Hint: $-(x^2+y^2) \leq 2xy \leq x^2+y^2$.

(5) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$.

(a) Write the definition of

(i) the function f is differentiable at the point $(x_0, y_0) \in \mathbb{R}^2$.

(ii) the first order partial derivatives of the function f at the point $(x_0, y_0) \in \mathbb{R}^2$

and point out the connection between the two definitions above.

(b) Let

$$f(x,y) = xye^{\left(\sqrt[5]{x} + \sqrt[3]{y}\right)}.$$

If possible, compute the gradient of f at $(0,0)$.

(6) Given the function $f(x,y) = \log\left(2x^2+y^2+1\right)$

(a) Compute the Taylor polynomial of order two of f around the point $(0,0)$.

(b) Using the section above, compute an approximate value of $f(0'1, -0'2)$, justifying your answer.

(c) Consider the function above $f : A \to \mathbb{R}$ defined on the set $A = [0,1] \times [0,1]$. Find, if they exist, the maximum and the minimum of f in A. Justify your answer.

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(7) Given the function $f(x,y) = x^4 - 6x^2 + y^2$

(a) Compute the critical points of f and classify them.

(b) Determine the open sets where f is strictly concave and/or convex.

(c) Argue whether f attains a global (or absolute) maximum and/or minimum on $D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 < x < 3\}$. If so, at what points?

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(8) Consider the function $f(x,y) = 2 + (x-y)^2$ and the set $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(a) Compute the Lagrange equations that determine the extreme points of f in A.

(b) Determine the points that satisfy the Lagrange equations and classify the extreme points of f in A.
