$\boxed{1}$

Consider the matrix

$$A = \begin{pmatrix} -2a & -6 & 6a \\ 0 & 4a & 4a \\ 0 & a & 4a \end{pmatrix}$$

where a is a real valued parameter.

(a) Find the eigenvalues and eigenvectors of A depending on the values of the parameter a.

(b) For which values of a is the matrix A diagonalizable? For these values, find matrices D diagonal and P such that $A = PDP^{-1}$ (Do not compute P^{-1}).

Solution:

1. The characteristic polynomial of A is

$$p(\lambda) = |A - \lambda I_3| = \begin{vmatrix} -2a-\lambda & -6 & 6a \\ 0 & 4a-\lambda & 4a \\ 0 & a & 4a-\lambda \end{vmatrix} = -(2a+\lambda)\begin{vmatrix} 4a-\lambda & 4a \\ a & 4a-\lambda \end{vmatrix}$$
$$= -(2a+\lambda)\left((4a-\lambda)^2 - 4a^2\right).$$

One root is $-2a$ and we get the rest from $(4a-\lambda)^2 - 4a^2 = 0$. Solving we get $\lambda = 4a \pm 2|a|$, that is, $2a$ and $6a$. Thus, the eigenvalues of A are $-2a, 2a, 6a$. For $a \neq 0$ the eigenvectors are

$$\begin{aligned} S(-2a) &= <(1,0,0)> \\ S(2a) &= <(6+3a, -4a, 2a)> \\ S(6a) &= <(-6+3a, 8a, 4a)> . \end{aligned}$$

For $a = 0$, the only eigenvalue of A is 0, and $S(0) = <(1,0,0)>$.

2. For $a \neq 0$, A admits three different eigenvalues, so it is diagonalizable. For $a = 0$ the matrix has only one eigenvector, thus it is not diagonalizable. For $a \neq 0$, $A = PDP^{-1}$ with

$$D = \begin{pmatrix} -2a & 0 & 0 \\ 0 & 2a & 0 \\ 0 & 0 & 6a \end{pmatrix} \quad P = \begin{pmatrix} 1 & 6+3a & -6+3a \\ 0 & -4a & 8a \\ 0 & 2a & 4a \end{pmatrix}.$$

2

A small beach is visited every summer by 900 families, but not every family remains there. Some of them abandon the beach if it does not show a clean aspect. Let y_t be the number of families on the beach at the beginning of summer t. Let x_t be the units of trash accumulated on the beach at the end of summer t. Then, out of the 900 families that visit the beach at the beginning of summer $t+1$, $\frac{x_t}{2}$ families decide not to spend time on it. On the other hand, beach work maintenance along year t diminishes trash for year $t+1$ to one half the existing trash at the beginning of the session. The trash produced is proportional to the number of families on the beach, $\frac{y_t}{8}$ units.

Summarizing, we have the following system that characterizes the evolution of trash/families on the beach

$$\begin{cases} x_{t+1} &= \frac{x_t}{2} + \frac{y_t}{8} \\ y_{t+1} &= 900 - \frac{x_t}{2}. \end{cases}$$

(a) Calculate the equilibrium values of trash and families on the beach. Suppose that in the summer t^*+1 there are no families on the beach, $y_{t^*+1} = 0$. How many units of trash there were at the end of the summer t^*?

(b) Study the stability of the equilibrium.

Solution:

1. The equilibrium is obtained by solving the system

$$\begin{cases} x &= \frac{x}{2} + \frac{y}{8} \\ y &= 900 - \frac{x}{2}. \end{cases}$$

We find $x^0 = 200$ units of trash and $y^0 = 800$ families. For the second question, from $0 = y_{t^*+1} = 900 - \frac{1}{2}x_{t^*}$ we easily get $x_{t^*} = 1800$.

2. To study the stability of the equilibrium, we compute the eigenvalues of the matrix of the system

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

The characteristic polynomial is $\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16}$, with only one root, $\lambda = \frac{1}{4}$. Since it is smaller than 1 in absolute value, the system is globally asymptotically stable.

$\boxed{3}$

Find the solution of $x_{t+2} + x_{t+1} - 2x_t = 3$ satisfying $x_0 = 3$, $x_1 = 1$.

Solution:

The roots of the characteristic equation $r^2+r-2=0$ are 1 and -2, thus the solution of the homogeneous equation is

$$C_1 + C_2(-2)^t.$$

Since 1 is a root of the characteristic equation and the independent term is a constant, we try the particular solution At, with A a suitable constant. Plugging it into the equation we get $A = 1$. Hence the general solution is

$$x_t = C_1 + C_2(-2)^t + t$$

Imposing the initial conditions we have the linear system

$$\begin{cases} 3 &= C_1 + C_2 \\ 1 &= C_1 - 2C_2 + 1 \end{cases}$$

Solving we get $C_1 = 2$ and $C_2 = 1$. Thus

$$x_t = 2 + (-2)^t + t.$$

$\boxed{4}$

Choose to solve one (and only one!) of the following equations

(a)

$$\frac{dx}{dt} = \frac{t}{x\sqrt{x^2+4}}, \quad \text{with } x(0) = 0.$$

(b) General solution of

$$\dot{x} + \frac{x}{1+t} = t.$$

Solution:

1. It is separable
$$x\sqrt{x^2+4}dx = tdt \Rightarrow \frac{1}{3}(x^2+4)^{\frac{3}{2}} = \frac{t^2}{2} + C.$$
Plugging in $t = 0$, $x(0) = 0$, we get $C = \frac{1}{3}4^{\frac{3}{2}} = \frac{1}{3}2^3 = \frac{8}{3}$. Hence solutions are implicitly given by
$$(x^2+4)^{\frac{3}{2}} = \frac{3t^2}{2} + 8.$$
(If we solve, we find $x(t) = \pm\sqrt{\left(\frac{3t^2}{2}+8\right)^{\frac{2}{3}} - 4}$).

2. It is a linear equation with non constant coefficients. The integrating factor is
$$\mu(t) = e^{\int \frac{dt}{1+t}} = e^{\ln(1+t)} = 1+t.$$
Multiplying the differential equation by the integrating factor we get $\frac{d(x\mu)}{dt} = t\mu$, hence integrating
$$x(t)(1+t) = \int t(1+t)dt = \frac{t^2}{2} + \frac{t^3}{3} + C,$$
where C is an arbitrary constant. Solving for $x(t)$ we find
$$x(t) = \frac{\frac{t^2}{2} + \frac{t^3}{3} + C}{1+t}.$$

$\boxed{5}$

Consider the differential equation,

$$(t + x)\ dt + at\, dx = 0, \quad t \geq 0, \quad a \neq 0$$

where $a \neq 0$ is a real valued parameter.

(a) For which values of the parameter a is the above differential exact?

(b) Suppose $a \neq 0$. Find the particular solution of the above differential equation which satisfies $x(1) = 0$.

Solution:

1. Note that
$$\frac{\partial}{\partial x}(t + x) = 1, \quad \frac{\partial at}{\partial t} = a$$
Hence, the differential equation is exact iff $a = 1$.

2. Since
$$\frac{1-a}{at}$$
depends only on t, the differential equation admits an integrating factor $\mu(t)$. This integrating factor satisfies that
$$\frac{\mu'}{\mu} = \frac{1-a}{a}\frac{1}{t}$$
That is,
$$\mu = t^{\frac{1-a}{a}}$$
is an integrating factor. The general solution of the DE is given implicitly by the equation
$$\frac{a}{1+a}t^{\frac{1+a}{a}} + axt^{\frac{1}{a}} = C.$$
Plugging in the values $t = 1$, $x = 0$ we obtain
$$C = \frac{a}{1+a}.$$

$\boxed{6}$

Consider the linear system of differential equationsof the linear system of differential equations

$$\begin{cases} \dot{x} & = -x - y, \\ \dot{y} & = ax - y. \end{cases}$$

(a) Study the stability and classify the equilibrium point when $a = 4$.

(b) Study the stability and classify the equilibrium point when $a = -4$.

Solution:

The characteristic equation is $(1 + \lambda)^2 + a = 0$.

1. When $a = 4$, the roots of the characteristic polynomial are complex with negative real part: $-1 \pm 2i$. Thus, the system is globally asymptotically stable. The equilibrium point $(0, 0)$ is an attractive spiral.

2. When $a = -4$, the roots of the characteristic polynomial are real $\lambda = -1 \pm 2$. One root is negative and the other is positive, thus the system is unstable. The equilibrium point $(0, 0)$ is a saddle point.