

1

Se considera la matriz

$$A = \begin{pmatrix} 3a & 0 & 0 \\ a & 2a & -1 \\ a^2 & -a^2 & 2a \end{pmatrix}$$

donde a es un parámetro.

- Calcular los autovalores y los autovectores de A en función de los valores de a .
- Discutir para qué valores de a la matriz A es diagonalizable. Para dichos valores de a , encontrar las matrices D y P tales que $A = PDP^{-1}$.
- Una vez sustituido $a = 1$ en la matriz A , hallar la solución general del siguiente sistema de ecuaciones en diferencias y estudiar su estabilidad.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Puede dejar la expresión de A^t como el producto de tres matrices.

Solución:

- The eigenvalues of A are $a, 3a, 3a$.
For $a \neq 0$ the eigenvectors are

$$\begin{aligned} S(a) &= \langle (0, 1, a) \rangle \\ S(3a) &= \langle (1, 0, a), (1, 1, 0) \rangle \end{aligned}$$

For $a = 0$ the matrix A is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues are $0, 0, 0$ and the eigenvectors are $S(0) = \langle (1, 0, 0), (0, 1, 0) \rangle$.

- For $a \neq 0$, the matrix A is diagonalizable, $A = PDP^{-1}$ with

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 3a & 0 \\ 0 & 0 & 3a \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & a & 0 \end{pmatrix}$$

For $a = 0$, the matrix is not diagonalizable.

- The matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

is obtained by plugging in the value $a = 1$ in the matrix A above. Thus

$$I - A = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

and we solve the system of equations

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

The solution is $x = -1, y = 0, z = 0$.

We have shown that

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^t & 0 \\ 0 & 0 & 3^t \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

Thus, the general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}^t \left(\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

with C_1, C_2, C_3 arbitrary real numbers. Since the eigenvalues are 1, 3, 3, the system is unstable.

2

Se considera la ecuación

$$x_{t+2} - x_{t+1} - 6x_t = 5 + 6t.$$

- (a) Encontrar la solución general.
 - (b) Encontrar la solución que satisface las condiciones iniciales $x_0 = 0$, $x_1 = 0$.
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Solución:

1. The characteristic polynomial is $r^2 - r - 6$ whose roots are $r_1 = 2$, $r_2 = -3$. Thus, the general solution of

$$x_{t+2} - x_{t+1} - 6x_t = 5 + 6t$$

is $C_1 2^t + C_2 (-3)^t$ with C_1, C_2 arbitrary real numbers. Now we look for a solution of the form $y_t = at + b$ of the equation $x_{t+2} - x_{t+1} - 6x_t = 5 + 6t$. We see that there is a solution with $a = b = -1$. Hence, the general solution is

$$x_t = C_1 2^t + C_2 (-3)^t - t - 1.$$

2. Plugging in the values $t = 0$ and $t = 1$ in the general solution we have that

$$\begin{aligned} 0 &= c_1 + c_2 - 1 \\ 0 &= 2c_1 - 3c_2 - 2 \end{aligned}$$

and solving this system of equations we obtain $c_1 = 1$, $c_2 = 0$ Hence, the solution is $x_t = 2^t - t - 1$.

3

- (a) Hallar la solución de la ecuación diferencial

$$t^2 y'(t) + y(t) = 1.$$

- (b) Hallar la solución de la ecuación diferencial

$$y'(t) = \frac{t^2}{y(t)}$$

que satisface la condición inicial $y(0) = 1$.

Solución:

1. The equation is linear. The general solution is

$$y(t) = C_1 e^{\frac{1}{t}} + 1.$$

2. The equation is separable. The general solution satisfies

$$y^2(t) = \frac{2}{3} t^3 + C.$$

The solution is $y(t) = \sqrt{\frac{2}{3} t^3 + 1}$.

4

- (a) Se considera la ecuación diferencial

$$y' = y^3 - y.$$

Encontrar y clasificar sus puntos de equilibrio.

- (b) Sea $y(t)$ la solución de la ecuación diferencial anterior, con dato inicial

$$y' = y^3 - y, \quad y(0) = \frac{1}{2}.$$

Hallar $\lim_{t \rightarrow \infty} y(t)$. Puede determinar si $y(t)$ es creciente o decreciente?

Solución:

1. Note that $f(y) = y^3 - y = y(y^2 - 1) = y(y - 1)(y + 1)$. Then the stationary points are $-1, 0, 1$. Note that $f < 0$ for $y < -1$, $f > 0$ for $-1 < y < 0$, $f < 0$ for $0 < y < 1$ and $f > 0$ for $y > 1$. Thus, we conclude that -1 and 1 are unstable and 0 is locally asymptotically stable.
2. $y(0) = 1/2 \in (0, 1)$ is in the region of stability of the differential equation, thus we see $\lim_{t \rightarrow -\infty} y(t) = 1$, $\lim_{t \rightarrow \infty} y(t) = 0$; moreover, $y(t)$ is decreasing, since $f(1/2) < 0$.

5

Se considera el siguiente sistema de ecuaciones diferenciales lineales

$$\begin{cases} \dot{x} &= x + by \\ \dot{y} &= bx + y \end{cases}$$

con $b \in \mathbb{R}$. Hallar los puntos de equilibrio y estudiar su estabilidad dependiendo de los valores del parámetro b .

Solución:

The equilibrium point is $(0, 0)$. The matrix associated to the linear system of differential equations is

$$A = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = 1 - b$ and $\lambda_2 = 1 + b$.

If $-1 < b < 1$, then $\lambda_1 > 0$ and $\lambda_2 > 0$ and $(0, 0)$ is an unstable node.

If $b = -1$ or $b = 1$, then the eigenvalues are 0 and 2, so $(0, 0)$ is an unstable node.

If $b < -1$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, or if $b > 1$, then $\lambda_1 < 0$ and $\lambda_2 > 0$, thus $(0, 0)$ is a saddle point.