

1

- (a) (5 puntos) Encontrar la solución general de la ecuación en diferencias siguiente.

$$x_{t+2} + 2x_{t+1} - 3x_t = 0.$$

- (b) (5 puntos) Encontrar la solución general de la ecuación en diferencias siguiente.

$$x_{t+2} + 2x_{t+1} - 3x_t = 16.$$

- (c) (5 puntos) Encontrar la solución particular de la ecuación en diferencias siguiente.

$$x_{t+2} + 2x_{t+1} - 3x_t = 16, \quad x_0 = 4, x_1 = 8.$$

Solución:

- (a) The characteristic equation is $r^2 + 2r - 3 = 0$ whose solutions are $r_1 = 1$, $r_2 = -3$. The general solution is

$$x_t = A + B(-3)^t.$$

- (b) Since $r_1 = 1$ is a root of the characteristic equation, we look for a particular solution of the form $y_t = At$. Plugging into the difference equation the values

$$y_t = At, \quad y_{t+1} = At + A, \quad y_{t+2} = At + 2A$$

and solving for A we obtain $A = 4$. The general solution is

$$x_t = 4t + A + B(-3)^t.$$

- (c) We have that the general solution is of the form $x_t = 4t + A + B(-3)^t$. Thus, $x_0 = A + B$, $x_1 = A - 3B + 4$. We solve the system

$$\begin{aligned} A + B &= 4 \\ A - 3B + 4 &= 8 \end{aligned}$$

and we obtain the solution $A = 4$, $B = 0$. Thus, the solution of the difference equation is

$$x_t = 4t + 4.$$

2

Se considera la matriz

$$A = \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

donde $a \in \mathbb{R}$.

- (a) (10 puntos) ¿Para qué valores del parámetro a es la matriz A diagonalizable?
- (b) (10 puntos) Para los valores del parámetro a para los que la matriz A es diagonalizable, escriba razonadamente su forma diagonal D y la matriz P .

Solución:

- (a) The characteristic polynomial of A is $(a - \lambda)(-\lambda(1 - \lambda) - 2)$, with roots

$$\lambda_1 = a, \lambda_2 = 2, \lambda_3 = -1.$$

If $a \neq 2$ and $a \neq -1$, then the polynomial has three different roots, thus the matrix A is diagonalizable. In the case $a = 2$, $\lambda_1 = \lambda_2 = 2$ is double. The rank of $A - 2I$ is

$$\text{rank}(A - 2I) = \text{rank} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix} = 2 \neq 1,$$

thus A is not diagonalizable. In the case $a = -1$, $\lambda_1 = \lambda_3 = -1$ is double. The rank of $A + I$ is

$$\text{rank}(A + I) = \text{rank} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} = 2 \neq 1,$$

thus A is not diagonalizable.

- (b) A is diagonalizable if and only if $a \neq 2$ and $a \neq -1$.

- To find an eigenvector associated to a , we solve $(A - aI)u = 0$, that is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -a & -2 \\ 0 & -1 & 1-a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and obtain $y = (1 - a)z$, $x = (2 + a(1 - a))z$.

- To find an eigenvector associated to 2, we solve $(A - 2I)u = 0$, that is

$$\begin{pmatrix} a-2 & 0 & 0 \\ 1 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and obtain $x = 0$, $y = -z$.

- To find an eigenvector associated to -1 , we solve $(A + I)u = 0$, or

$$\begin{pmatrix} a+1 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and obtain $x = 0$, $y = 2z$.

Suitable matrices D and P are

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 2 + a(1 - a) & 0 & 0 \\ 1 - a & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

respectively.

3

Se considera el siguiente sistema no homogéneo de ecuaciones en diferencias lineales, $X_{t+1} = AX_t + B$, donde la matriz de coeficientes es la matriz A del Problema 2 anterior para $a = \frac{1}{2}$, y que está dado por

$$\begin{aligned}x_{t+1} &= \frac{x_t}{2} + 1 \\y_{t+1} &= x_t - 2z_t - 2 \\z_{t+1} &= -y_t + z_t + 2\end{aligned}$$

- (a) (10 puntos) Encontrar el punto de equilibrio del sistema de ecuaciones en diferencias y estudiar su estabilidad.
 (b) (5 puntos) Calcular la solución general del sistema de ecuaciones en diferencias.
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Solución:

- (a) The equilibrium point is found by solving

$$\begin{cases} x &= \frac{x}{2} + 1 \\ y &= x - 2z - 2 \\ z &= -y + z + 2 \end{cases},$$

which solution is $x^0 = 2$, $y^0 = 2$, $z^0 = -1$. By Problem 2, the eigenvalues of the matrix of the system are $\frac{1}{2}$, 2 and -1 , thus the system is unstable. Since one of the eigenvalues is smaller than 1 in absolute value, there are initial conditions for which the solutions converges to the equilibrium point, which is a saddle point.

1. We know that the system is diagonalizable and we have found the eigenvalues and eigenvectors in Problem 2.
2. The general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1 2^{-t} \begin{pmatrix} 2 + \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix} + C_2 2^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 (-1)^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},$$

where C_1 , C_2 and C_3 are constants.

4

- (a) (10 puntos) Encontrar la solución general de la ecuación diferencial

$$\ddot{x} + 4x = 16te^{2t}.$$

- (b) (5 puntos) Encontrar la solución $x(t)$ de la ecuación diferencial anterior que satisface las condiciones iniciales

$$x(0) = 0, \quad \dot{x}(0) = -2.$$

Solución:

- (a) The characteristic equation is $r^2 + 4 = 0$ whose roots are $\pm 2i$. Hence, the general solution of the associated homogeneous equation is

$$x^h(t) = C_1 \sin(2t) + C_2 \cos(2t).$$

We look now for a particular solution of the form

$$x(t) = (At + B)e^{2t}$$

Thus,

$$x'(t) = e^{2t}(2At + A + 2B)$$

$$x''(t) = 4e^{2t}(At + A + B)$$

$$x'' + 4x = 4e^{2t}(2At + A + 2B)$$

and we obtain $4A + 8B = 0$ and $8A = 16$. Solving, we find $A = 2$, $B = -1$. Hence, the general solution is

$$x^g(t) = C_1 \sin(2t) + C_2 \cos(2t) + (2t - 1)e^{2t}$$

- (b) Note that plugging the values $x^g(0) = C_1 - 1 = 0$ and $\dot{x}^g(0) = 2C_2 = -2$ into the general solution, we get $C_1 = 1$ and $C_2 = -1$. Hence, the solution is

$$x^p(t) = \sin(2t) - \cos(2t) + (2t - 1)e^{2t}$$

5

Se considera el siguiente sistema de ecuaciones diferenciales lineales.

$$\begin{aligned}x' &= -x + 3y - 4 \\y' &= 3x - y + 1\end{aligned}$$

- (a) (10 puntos) ¿Cuál es el punto de equilibrio del sistema? ¿Es estable? En el caso de que el punto de equilibrio no sea estable, estudiar la posible existencia de una variedad estable y encontrarla.
- (b) (5 puntos) Calcular la solución general del sistema.

Solución:

- (a) The equilibrium point is $(1, 2)$. The eigenvalues are 2 and -4 , so the equilibrium point is a saddle point. The stable manifold is $S(-4) = \langle (1, -1) \rangle$.
- (b) $S(2) = \langle (1, 1) \rangle$. The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

6

Se considera la siguiente ecuación diferencial

$$\left(\frac{y}{x}\right) dx + \left(1 + \frac{e^y}{x}\right) dy = 0.$$

- (a) (5 puntos) Probar que admite un factor integrante de la forma $\mu(x)$.
- (b) (10 puntos) Calcular la solución general de la ecuación diferencial anterior.
- (c) (5 puntos) Calcular la solución del siguiente problema de valores iniciales.

$$\left(\frac{y}{x}\right) dx + \left(1 + \frac{e^y}{x}\right) dy = 0, \quad y(1) = 0.$$

Solución:

- (a) Letting $P(x, y) = \frac{y}{x}$ and $Q(x, y) = 1 + \frac{e^y}{x}$, we have that

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{\frac{1}{x} - \left(\frac{-e^y}{x^2}\right)}{1 + \frac{e^y}{x}} = \frac{1}{x} \frac{\left(1 + \frac{e^y}{x}\right)}{\left(1 + \frac{e^y}{x}\right)} = \frac{1}{x},$$

is independent of y . Hence, the DE has an integrating factor $\mu(x)$.

- (b) Let

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

Multiplying the DE by x , it becomes exact. Let

$$V(x, y) = \int xQ(x, y)dy = \int (x + e^y) dy = xy + e^y + g(x).$$

Since $\frac{\partial V}{\partial x} = xP = y$, we find $y + g'(x) = y$, hence $g(x) = 0$. The general solution of the DE is then

$$xy + e^y = C.$$

- (c) Plugging $y = 0$ and $x = 1$ into the general solution found in part (b) above, we get $C = 1$. Hence the particular solution is

$$xy + e^y = 1.$$