TEMAS DE MATEMÁTICAS AVANZADAS PARA LA ECONOMÍA

Hoja 8. Ecuaciones Diferenciales (4)

Soluciones

8-1. *Clasificar el punto de equilibrio* $(0,0)$ *de los sistemas siguientes, en términos del parámetro* α.

a) $\dot{X} = \begin{pmatrix} \alpha & 0 \\ 6 & 2\alpha \end{pmatrix} X, \quad (\alpha \neq 0).$

b) $\dot{X} = \begin{pmatrix} \alpha & -3 \\ 3 & \alpha \end{pmatrix} X.$

Solución:

a) $p_A(\lambda) = (\alpha - \lambda)(2\alpha - \lambda) = 0 \Leftrightarrow \lambda_{1,2} = \alpha, 2\alpha$. Hence, the system is g.a.s. when $\alpha < 0$ ($(0,0)$ stable node) and unstable when $\alpha > 0$ ($(0,0)$ unstable node).

b) $p_A(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + 9 = 0 \Leftrightarrow \lambda_{1,2} = \alpha \pm i3$.

1) $\alpha < 0$ attracting spiral; the system is g.a.s.,
2) $\alpha = 0$ center; the system is stable, but not g.a.s.,
3) $\alpha > 0$ repulsive spiral; unstable system.

8-2. *Determinar y clasificar el punto de equilibrio de los siguientes sistemas. En caso de aparecer un punto de silla, encontrar la variedad estable.*

a) $\dot{X} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X + \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$

b) $\dot{X} = \begin{pmatrix} 2 & -5 \\ 5 & -6 \end{pmatrix} X + \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$

Solución:

a) $(x^0, y^0) = (-1, 3)$ is a saddle. The stable manifold is the eigenspace associated to the negative eigenvalue, $S(-1) = \{2x + y = 0\}$. In this example where the equilibrium point is not $(0,0)$, to find the initial conditions from which the system converges to $(-1,3)$ it is needed to make a parallel displacement of the stable manifold to the line passing through $(-1,3)$, i.e.

$$2(x - (-1)) + y - 3 = 0.$$

Thus, initial conditions (x_0, y_0) satisfying $2x_0 + y_0 = -1$ generate trajectories $(x(t), y(t))$ such that $x(t) \to -1$ and $y(t) \to 3$ when $t \to \infty$.

b) $(x^0, y^0) = (3, 1)$ and $\lambda_{1,2} = -2 \pm i3$. Since the real part is negative, the equilibrium point is an attractor spiral.

8-3. *Estudiar la estabilidad de los siguientes sistemas.*

$$a) \left\{ \begin{array}{ll} \dot{x} & = e^x - 1, \\ \dot{y} & = y e^x. \end{array} \right. \qquad b) \left\{ \begin{array}{ll} \dot{x} & = x^3 + 3x^2 y + y, \\ \dot{y} & = x(1 + y^2). \end{array} \right.$$

Solución: Both are nonlinear systems, so we approach the systems by linear systems. In case a) the only equilibrium point is $(0,0)$. We compute the derivatives

$$\frac{\partial \dot{x}}{\partial x} = e^x, \ \frac{\partial \dot{x}}{\partial y} = 0, \ \frac{\partial \dot{y}}{\partial x} = y e^x, \ \frac{\partial \dot{y}}{\partial y} = e^x.$$

The Jacobian matrix at $(0,0)$ is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, the system is unstable, hence the nonlinear system is also unstable.

Again $(0,0)$ is the only equilibrium point of system b) (notices that the factor $1+y^2$ never vanishes). We compute the derivatives

$$\frac{\partial \dot{x}}{\partial x} = 3x^2 + 6xy, \quad \frac{\partial \dot{x}}{\partial y} = 3x^2 + 1, \quad \frac{\partial \dot{y}}{\partial x} = 1 + y^2, \quad \frac{\partial \dot{y}}{\partial y} = 2xy.$$

The Jacobian matrix is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues ± 1, so the system is unstable, and the equilibrium point is locally a saddle point.

8-4. *The model of Obst*[1] *of monetary policy in the presence of an inflation adjustment mechanism is as follows. The quotient* M_d/M_s *(money demand/money supply), is denoted by* μ; $p = \dot{P}/P$ *is the inflation rate (*P *is the price level of the economy);* $q = \dot{Q}/Q$ *the constant (exogenous) rate of growth of GDP,* Q, *and* $m = \dot{M}_s/M_s$ *the monetary expansion rate. The evolution of* p *follows the Walrasian adjustment mechanism*

$$\dot{p} = h(1-\mu), \qquad 0 < h < 1 \ \textit{a parameter.}$$

Hence an excess in the monetary supply $M_s > M_d$, *leads to a positive increment in the inflation rate. To stipulate the time evolution of* μ *we consider the following assumption: monetary demand is proportional to GDP in nominal terms, that is,*

$$M_d = aPQ, \qquad a > 0 \ \textit{constant,}$$

hence

$$\mu = a\frac{PQ}{M_s}.$$

Taking logarithms

$$\ln \mu = \ln a + \ln P + \ln Q - \ln M_s,$$

and taking the derivative with respect to time we get

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} - \frac{\dot{M}_s}{M_s} = p + q - m.$$

Hence, the system of ODEs in the model of Obst is

$$\dot{p} = h(1-\mu),$$
$$\dot{\mu} = (p + q - m)\mu.$$

The exercise studies the effect of the monetary policy chosen by the central bank, given by m.

a) *Suppose that* $m = \overline{m}$ *is constant (*exogenous and constant monetary expansion rate*) and that* $\overline{m} > q$. *Show that the system has a center.*

b) *Suppose that* $m = \overline{m} - \alpha p$ *with* $\alpha > 0$ *(*countercyclical conventional monetary policy*) and* $\overline{m} > q$. *Show by means of the phase portrait that the qualitative behavior of the system is similar to (a) above.*

c) *Suppose that* $m = \overline{m} - \alpha\dot{p}$ *(*Obst's Rule*) with* $\alpha > 0$ *and* $\overline{m} > q$. *Prove that for some values of* α *the system has a spiral attractor.*

d) *What do you think about the stabilization properties of the countercyclical rule and Obst's Rule?*

Solución:

a) In the case $m = \overline{m}$ constant, the equilibrium point is $(\overline{m} - q, 1)$. In equilibrium, money demand equals supply and the inflation rate is given by $\overline{m} - q$. The question is if a constant monetary policy leads to stabilization of inflation and money demand/supply.
The Jacobian matrix at the equilibrium point is

$$\begin{pmatrix} 0 & -h \\ \mu & p + q - \overline{m} \end{pmatrix}_{p = \overline{m} - q, \mu = 1} = \begin{pmatrix} 0 & -h \\ 1 & 0 \end{pmatrix}.$$

[1] N. P. Obst (1978) "Stabilization policy with an inflation adjustment mechanism". *Quarterly Journal of Economics*, May, pp. 355–359.

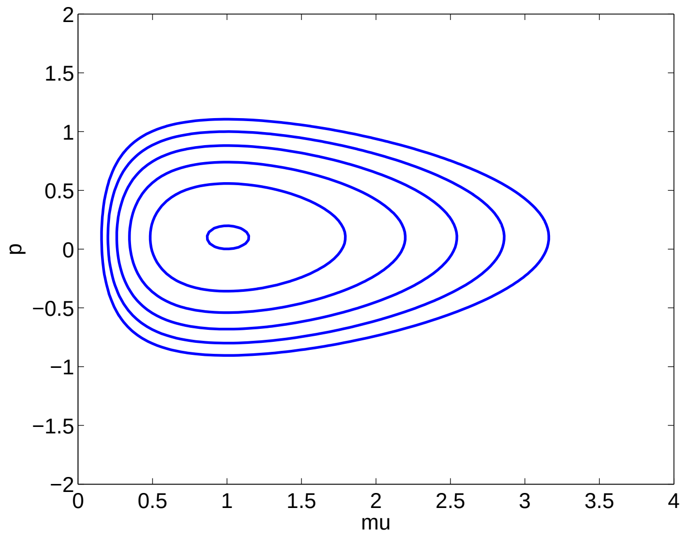
The eigenvalues are pure imaginary, hence the equilibrium point is a center of the linearized system. We cannot infer the same behavior for the nonlinear system in this case, see Theorem 6.35. However, the system is explicitly solvable, because

$$\frac{\dot{p}}{\dot{\mu}} = \frac{dp}{d\mu} = \left(\frac{h}{p+q-\overline{m}}\right)\left(\frac{1-\mu}{\mu}\right)$$

is a separable ODE. Integrating in the usual way, we get

$$\frac{p^2}{2} - (\overline{m}-q)p = h(\ln\mu - \mu) + C,$$

where C is constant. The solution curves are shown below. They surround the equilibrium point, thus the model has an oscillatory behavior around it. The equilibrium is a center



b) Consider now $m(p) = \overline{m} - \alpha p$. This policy is called countercyclical since the monetary expansion decrease if the inflation rate increases. The equilibrium point now is $(\frac{\overline{m}-q}{1+\alpha}, 1)$. The inflation rate is smaller than with m constant. However, the situation is much as above. The equilibrium is again a center, both for the linear and the nonlinear systems. The phase portrait is qualitatively similar to figure above.

c) Consider now Obst' Rule, $m(\dot{p}) = \overline{m} - \alpha\dot{p}$. Now the monetary expansion rate responds not only with respect to changes in the inflation rate, but in its instantaneous variation (a more sensitive rule). We get the system

$$\dot{p} = h(1-\mu),$$
$$\dot{\mu} = (p + q - \overline{m} + \alpha h(1-\mu))\mu.$$

The equilibrium point is again $(\overline{m} - q, 1)$ and the Jacobian matrix at it is

$$\begin{pmatrix} 0 & -h \\ 1 & -\alpha h \end{pmatrix}.$$

It is easy to check that the eigenvalues

$$\lambda_{1,2} = \frac{-\alpha h \pm \sqrt{\alpha^2 h^2 - 4h}}{2}$$

have negative real part for any $h, \alpha > 0$. Hence, the nonlinear system is locally asymptotically stable. If $\alpha^2 h^2 - 4h < 0$ the equilibrium is a stable spiral point.

d) Monetary policies based in exogenous monetary expansion or in countercyclical specifications do not stabilize inflation and money demand. However, Obst's Rule does.