UC3M
TOPICS Advanced Mathematics for Economics
Final January 15th, 2024 **Answers**

1

(a) (5 points) Determine the general solution of the homogenous difference equation

$$x_{t+2} - x_{t+1} - 2x_t = 0.$$

(b) (5 points) Determine the general solution of the non homogenous difference equation

$$x_{t+2} - x_{t+1} - 2x_t = 1 + 2^t.$$

(c) (5 points) Determine the solution of the initial value problem

$$x_{t+2} - x_{t+1} - 2x_t = 1 + 2^t, \qquad x_0 = 0,\ x_1 = 2.$$

Solution:

(a) The characteristic equation is $r^2 - r - 2 = 0$, with solutions -1 and 2, hence the general solution is $C_1(-1)^t + C_2 2^t$.

(b) Consider the particular solution $A + Bt2^t$ and determine the unknown constants by plugging it into the difference equation

$$A + B(t+2)2^{t+2} - A - B(t+1)2^{t+1} - 2(A + Bt2^t) = 1 + 2^t.$$

Equating e ual expressions at the left and at the right hand side we obtain $0 = 0$ for $t2^t$, $8B - 2B = 1$ for 2^t, and $A - A - 2A = 1$ for the independent term. The particular solution is thus $-\frac{1}{2} + \frac{1}{6}t2^t$ and the general solution is

$$x_t = C_1(-1)^t + C_2 2^t - \frac{1}{2} + \frac{1}{6}t2^t.$$

(c) The constants C_1 and C_2 satisfy

$$0 = x_0 = C_1 + C_2 - \frac{1}{2}$$

$$2 = x_1 = -C_1 + 2C_2 - \frac{1}{2} + \frac{1}{3}$$

The solution is $C_2 = \frac{8}{9}$ and $C_1 = -\frac{7}{18}$.
The solution asked for is $x_t = -\frac{7}{18}(-1)^t + \frac{8}{9}2^t - \frac{1}{2} + \frac{1}{6}t2^t$.

2

Let the matrix

$$A = \begin{pmatrix} -a^2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & -a^2 \end{pmatrix},$$

where a is a parameter.

(a) (5 points) Determine the eigenvalues of A. For which values of the parameter a is the matrix A diagonalizable?

(b) (5 points) For the values of a found in part (a) above, determine a diagonal matrix D and an inversible matrix P such that $P^{-1}AP = D$.

Solution:

(a) Since the matrix is triangular, the diagonal elements of A are the eigenvalues: $\lambda_1 = -a^2$ double and $\lambda_2 = 2$ simple. Note that $\lambda_1 \neq \lambda_2$ for all a.

A is diagonalizable iff the rank of $(A - (-a^2 I)_3) = (A + a^2 I_3)$ is one.

$$A + a^2 I_3 = \begin{pmatrix} 0 & 3 & 2 \\ 0 & \frac{1}{2} + a^2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank one iff $\begin{vmatrix} 3 & 2 \\ \frac{1}{2} + a^2 & 1 \end{vmatrix} = 3 - 2(\frac{1}{2} + a^2) = 0$, that is, iff $a^2 = 1$.

Hence, A is diagonalizable iff $a^2 = 1$.

(b) Let $a^2 = 1$.

$S(-a^2) = S(-1)$:

$$\begin{pmatrix} 0 & 3 & 2 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solutions: $(x, -\frac{2}{3}z, z)$. We choose $(1, 0, 0)$ and $(0, -\frac{2}{3}, 1)$.

$S(2)$:

$$\begin{pmatrix} -\frac{3}{2} & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solutions: $(2y, y, 0)$. We choose $(2, 1, 0)$.

Letting P

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -\frac{2}{3} & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

we take

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

3

Consider the following system of linear difference equations.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} -a^2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & -a^2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

with a a parameter. Note that the matrix of the system is the matrix A of Problem 2 above.

(a) (5 points) Find the stationary or equilibrium point of the system. For the values of a for which the matrix of the system is diagonalizable (they were found in Problem 2 above), study whether the fixed point is asymptotically stable.

(b) (5 points) For the values of a for which the matrix of the system is diagonalizable (they were found in Problem 2 above), determine the general solution of the system.

Solution:

(a) Steady states are solutions of the system

$$\begin{cases} x = -a^2 x + 3y + 2z \\ y = +\frac{1}{2} y + z + 1 \\ z = -a^2 z \end{cases}$$

The steady state is $(\frac{6}{1+a^2}, 2, 0)$. According to Problem 2, the matrix of coefficients of the system is diagonalizable iff $a^2 = 1$. Thne steady state is $(3, 2, 0)$. The eigenvalues are -1 double and $\frac{1}{2}$ simple. The steady state is not asymptotically stable, but it is a saddle point.

(b) Using the results of Problem 2, and part (a) above, the general solution of the system is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1(-1)^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2(-1)^t \begin{pmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{pmatrix} + C_3 2^t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$$

4

Consider the ODE

$$t^3x' + 2t^2x = 1,$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution of the initial value problem

$$t^3x' + 2t^2x = 1, \quad x(-1) = 2$$

What is the interval of definition of the solution?

Solution:

The general solution is

$$\frac{\ln|t|}{t^2} + \frac{c_1}{t^2}$$

The solution of the initial value problem

$$t^3x' + 2t^2x = 1, \quad x(-1) = 2$$

is

$$\frac{\ln(-t)}{t^2} + \frac{2}{t^2}$$

the domain of definition is $(-\infty, 0)$.

5

(a) (5 points) Find the general solution of the ODE

$$x'' + 6x' + 9x = 0$$

(b) (5 points) Find the general solution of the ODE

$$x'' + 6x' + 9x = 6e^{-3t}$$

(c) (5 points) Find the solution of the ODE of the initial value problem

$$x'' + 6x' + 9x = 6e^{-3t}, \quad x(0) = 1 \quad x'(0) = -1$$

Solution:

The general solution of $x'' + 6x' + 9x = 6e^{-3t}$ is

$$Ae^{-3t} + Be^{-3t}t + 3e^{-3t}t^2$$

For the given initial conditions, the solution is

$$e^{-3t}\left(3t^2 + 2t + 1\right)$$

6

Consider the following system of ODE's

$$\begin{cases} x' &= x^2y - \frac{x^2}{16} - 25y + \frac{25}{16} \\ y' &= xy - 4y \end{cases}$$

(a) (5 points) Determine the stationary points.

(b) (5 points) Determine the stability of the stationary points.

Solution:

The stationary points are the solutions of the following system of equations

$$\begin{cases} 0 &= x^2y - \frac{x^2}{16} - 25y + \frac{25}{16} \\ 0 &= xy - 4y \end{cases}$$

The second equation may be written as $y(x-4) = 0$. The solutions are $x = 4$ and $y = 0$.Plugging this into the first equation we obtain the stationary points are $(-5, 0)$, $(5, 0)$ and $\left(4, \frac{1}{16}\right)$. The Jacobian matrix of the system is

$$J(x, y) = \begin{pmatrix} 2xy - \frac{x}{8} & x^2 - 25 \\ y & x - 4 \end{pmatrix}$$

We see that

- $J(-5, 0) = \begin{pmatrix} \frac{5}{8} & 0 \\ 0 & -9 \end{pmatrix}$. The eigenvalues are $-9, \frac{5}{8}$ and $(-5, 0)$ is unstable. It is a saddle point.
- $J(5, 0) = \begin{pmatrix} -\frac{5}{8} & 0 \\ 0 & 1 \end{pmatrix}$. The eigenvalues are $-\frac{5}{8}, 1$ and $(5, 0)$ is unstable. It is a saddle point.
- $J\left(4, \frac{1}{16}\right) \begin{pmatrix} 0 & -9 \\ \frac{1}{16} & 0 \end{pmatrix}$. The eigenvalues are $\pm\frac{3i}{4}$ and $\left(4, \frac{1}{16}\right)$ is a l.a.s. It is a spiral point.