UC3M
TOPICS Advanced Mathematics for Economics
Final January 15th, 2024

Niu: ______________ Group: ______

Name: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	15	10	10	10	15	10	70
Score:							

Instructions:

- **DURATION OF THE EXAM: 2h**.
- Calculators are **NOT** allowed.
- **DO NOT UNSTAPLE** the exam.
- You must show a valid ID to the professor if required.
- Read the exam carefully. The exam has 6 questions, for a total of 70 points.
- Remember that for a complex number $z = a + ib$, the module is $\rho = |z| = \sqrt{a^2 + b^2}$, and the argument θ is the angle such that $\tan\theta = b/a$.

  Table of usual trigonometric values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

$\boxed{1}$

(a) (5 points) Determine the general solution of the homogenous difference equation

$$x_{t+2} - x_{t+1} - 2x_t = 0.$$

(b) (5 points) Determine the general solution of the non homogenous difference equation

$$x_{t+2} - x_{t+1} - 2x_t = 1 + 2^t.$$

(c) (5 points) Determine the solution of the initial value problem

$$x_{t+2} - x_{t+1} - 2x_t = 1 + 2^t, \qquad x_0 = 0,\ x_1 = 2.$$

$\boxed{2}$

Let the matrix

$$A = \begin{pmatrix} -a^2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & -a^2 \end{pmatrix},$$

where a is a parameter.

(a) (5 points) Determine the eigenvalues of A. For which values of the parameter a is the matrix A diagonalizable?

(b) (5 points) For the values of a found in part (a) above, determine a diagonal matrix D and an inversible matrix P such that $P^{-1}AP = D$.

$\boxed{3}$

Consider the following system of linear difference equations.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} -a^2 & 3 & 2 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & -a^2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

with a a parameter. Note that the matrix of the system is the matrix A of Problem 2 above.

(a) (5 points) Find the stationary or equilibrium point of the system. For the values of a for which the matrix of the system is diagonalizable (they were found in Problem 2 above), study whether the fixed point is asymptotically stable.

(b) (5 points) For the values of a for which the matrix of the system is diagonalizable (they were found in Problem 2 above), determine the general solution of the system.

$\boxed{4}$

Consider the ODE

$$t^3x' + 2t^2x = 1,$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution of the initial value problem

$$t^3x' + 2t^2x = 1, \quad x(-1) = 2$$

What is the interval of definition of the solution?

$\boxed{5}$

(a) (5 points) Find the general solution of the ODE

$$x'' + 6x' + 9x = 0$$

(b) (5 points) Find the general solution of the ODE

$$x'' + 6x' + 9x = 6e^{-3t}$$

(c) (5 points) Find the solution of the ODE of the initial value problem

$$x'' + 6x' + 9x = 6e^{-3t}, \quad x(0) = 1 \quad x'(0) = -1$$

$\boxed{6}$

Consider the following system of ODE's

$$\begin{cases} x' &= x^2 y - \frac{x^2}{16} - 25y + \frac{25}{16} \\ y' &= xy - 4y \end{cases}$$

(a) (5 points) Determine the stationary points.

(b) (5 points) Determine the stability of the stationary points.
