TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

Sheet 8. Differential Equations (4)

Solutions

8-1. *Classify the equilibrium point* $(0,0)$ *of the following systems, in terms of the parameter* α.

(a) $\dot{X} = \begin{pmatrix} \alpha & 0 \\ 6 & 2\alpha \end{pmatrix} X, \quad (\alpha \neq 0).$

(b) $\dot{X} = \begin{pmatrix} \alpha & -3 \\ 3 & \alpha \end{pmatrix} X.$

Solution:

(a) $p_A(\lambda) = (\alpha - \lambda)(2\alpha - \lambda) = 0 \Leftrightarrow \lambda_{1,2} = \alpha, 2\alpha$. Hence, the system is g.a.s. when $\alpha < 0$ ($(0,0)$ stable node) and unstable when $\alpha > 0$ ($(0,0)$ unstable node).

(b) $p_A(\lambda) = \lambda^2 - 2\alpha\lambda + \alpha^2 + 9 = 0 \Leftrightarrow \lambda_{1,2} = \alpha \pm i3$.
- (i) $\alpha < 0$ attracting spiral; the system is g.a.s.,
- (ii) $\alpha = 0$ center; the system is stable, but not g.a.s.,
- (iii) $\alpha > 0$ repulsive spiral; unstable system.

8-2. *Find and classify the equilibrium point of the following systems. In the case of a saddle, find the stable manifold.*

(a) $\dot{X} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X + \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$

(b) $\dot{X} = \begin{pmatrix} 2 & -5 \\ 5 & -6 \end{pmatrix} X + \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$

Solution:

(a) $(x^0, y^0) = (-1, 3)$ is a saddle. The stable manifold is the eigenspace associated to the negative eigenvalue, $S(-1) = \{2x + y = 0\}$. In this example where the equilibrium point is not $(0,0)$, to find the initial conditions from which the system converges to $(-1,3)$ it is needed to make a parallel displacement of the stable manifold to the line passing through $(-1,3)$, i.e.

$$2(x - (-1)) + y - 3 = 0.$$

Thus, initial conditions (x_0, y_0) satisfying $2x_0 + y_0 = -1$ generate trajectories $(x(t), y(t))$ such that $x(t) \to -1$ and $y(t) \to 3$ when $t \to \infty$.

(b) $(x^0, y^0) = (3, 1)$ and $\lambda_{1,2} = -2 \pm i3$. Since the real part is negative, the equilibrium point is an attractor spiral.

8-3. *Study the stability of the following systems.*

$$a)\ \begin{cases} \dot{x} &= e^x - 1, \\ \dot{y} &= ye^x. \end{cases} \qquad b)\ \begin{cases} \dot{x} &= x^3 + 3x^2 y + y, \\ \dot{y} &= x(1 + y^2). \end{cases}$$

Solution: Both are nonlinear systems, so we approach the systems by linear systems. In case a) the only equilibrium point is $(0,0)$. We compute the derivatives

$$\frac{\partial \dot{x}}{\partial x} = e^x, \ \frac{\partial \dot{x}}{\partial y} = 0, \ \frac{\partial \dot{y}}{\partial x} = ye^x, \ \frac{\partial \dot{y}}{\partial y} = e^x.$$

The Jacobian matrix at $(0,0)$ is $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly, the system is unstable, hence the nonlinear system is also unstable.

Again $(0,0)$ is the only equilibrium point of system b) (notices that the factor $1+y^2$ never vanishes). We compute the derivatives

$$\frac{\partial \dot{x}}{\partial x} = 3x^2 + 6xy, \ \frac{\partial \dot{x}}{\partial y} = 3x^2 + 1, \ \frac{\partial \dot{y}}{\partial x} = 1 + y^2, \ \frac{\partial \dot{y}}{\partial y} = 2xy.$$

The Jacobian matrix is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with eigenvalues ± 1, so the system is unstable, and the equilibrium point is locally a saddle point.

8-4. *The model of Obst*[1] *of monetary policy in the presence of an inflation adjustment mechanism is as follows. The quotient M_d/M_s (money demand/money supply), is denoted by μ; $p = \dot{P}/P$ is the inflation rate (P is the price level of the economy); $q = \dot{Q}/Q$ the constant (exogenous) rate of growth of GDP, Q, and $m = \dot{M}_s/M_s$ the monetary expansion rate. The evolution of p follows the Walrasian adjustment mechanism*

$$\dot{p} = h(1-\mu), \qquad 0 < h < 1 \text{ a parameter.}$$

Hence an excess in the monetary supply $M_s > M_d$, leads to a positive increment in the inflation rate. To stipulate the time evolution of μ we consider the following assumption: monetary demand is proportional to GDP in nominal terms, that is,

$$M_d = aPQ, \qquad a > 0 \text{ constant,}$$

hence

$$\mu = a\frac{PQ}{M_s}.$$

Taking logarithms

$$\ln \mu = \ln a + \ln P + \ln Q - \ln M_s,$$

and taking the derivative with respect to time we get

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} - \frac{\dot{M}_s}{M_s} = p + q - m.$$

Hence, the system of ODEs in the model of Obst is

$$\begin{aligned} \dot{p} &= h(1-\mu), \\ \dot{\mu} &= (p+q-m)\mu. \end{aligned}$$

The exercise studies the effect of the monetary policy chosen by the central bank, given by m.

(a) *Suppose that $m = \overline{m}$ is constant (*exogenous and constant monetary expansion rate*) and that $\overline{m} > q$. Show that the system has a center.*

(b) *Suppose that $m = \overline{m} - \alpha p$ with $\alpha > 0$ (*countercyclical conventional monetary policy*) and $\overline{m} > q$. Show by means of the phase portrait that the qualitative behavior of the system is similar to (a) above.*

(c) *Suppose that $m = \overline{m} - \alpha\dot{p}$ (*Obst's Rule*) with $\alpha > 0$ and $\overline{m} > q$. Prove that for some values of α the system has a spiral attractor.*

(d) *What do you think about the stabilization properties of the countercyclical rule and Obst's Rule?*

Solution:

(a) In the case $m = \overline{m}$ constant, the equilibrium point is $(\overline{m} - q, 1)$. In equilibrium, money demand equals supply and the inflation rate is given by $\overline{m} - q$. The question is if a constant monetary policy leads to stabilization of inflation and money demand/supply.
The Jacobian matrix at the equilibrium point is

$$\begin{pmatrix} 0 & -h \\ \mu & p + q - \overline{m} \end{pmatrix}_{p = \overline{m} - q, \mu = 1} = \begin{pmatrix} 0 & -h \\ 1 & 0 \end{pmatrix}.$$

[1] N. P. Obst (1978) "Stabilization policy with an inflation adjustment mechanism". *Quarterly Journal of Economics*, May, pp. 355–359.

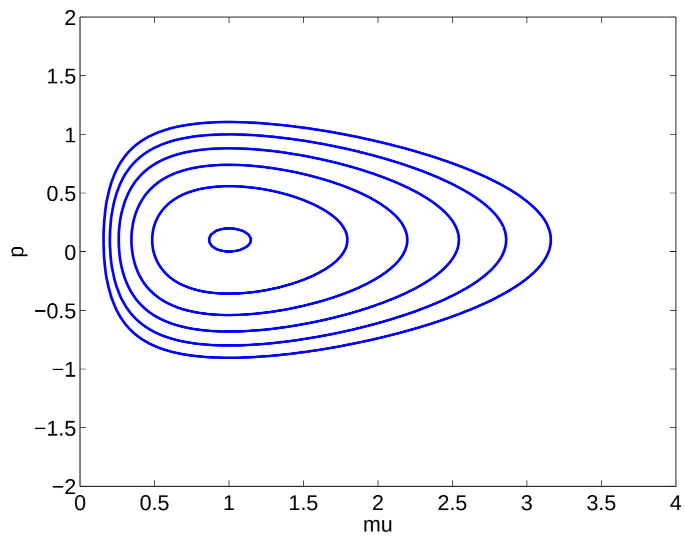
The eigenvalues are pure imaginary, hence the equilibrium point is a center of the linearized system. We cannot infer the same behavior for the nonlinear system in this case, see Theorem 6.35. However, the system is explicitly solvable, because

$$\frac{\dot{p}}{\dot{\mu}} = \frac{dp}{d\mu} = \left(\frac{h}{p+q-\overline{m}}\right)\left(\frac{1-\mu}{\mu}\right)$$

is a separable ODE. Integrating in the usual way, we get

$$\frac{p^2}{2} - (\overline{m}-q)p = h(\ln\mu - \mu) + C,$$

where C is constant. The solution curves are shown below. They surround the equilibrium point, thus the model has an oscillatory behavior around it. The equilibrium is a center



(b) Consider now $m(p) = \overline{m} - \alpha p$. This policy is called countercyclical since the monetary expansion decrease if the inflation rate increases. The equilibrium point now is $(\frac{\overline{m}-q}{1+\alpha}, 1)$. The inflation rate is smaller than with m constant. However, the situation is much as above. The equilibrium is again a center, both for the linear and the nonlinear systems. The phase portrait is qualitatively similar to figure above.

(c) Consider now Obst' Rule, $m(\dot{p}) = \overline{m} - \alpha\dot{p}$. Now the monetary expansion rate responds not only with respect to changes in the inflation rate, but in its instantaneous variation (a more sensitive rule). We get the system

$$\begin{aligned}\dot{p} &= h(1-\mu),\\ \dot{\mu} &= (p+q-\overline{m}+\alpha h(1-\mu))\mu.\end{aligned}$$

The equilibrium point is again $(\overline{m}-q, 1)$ and the Jacobian matrix at it is

$$\begin{pmatrix} 0 & -h \\ 1 & -\alpha h \end{pmatrix}.$$

It is easy to check that the eigenvalues

$$\lambda_{1,2} = \frac{-\alpha h \pm \sqrt{\alpha^2 h^2 - 4h}}{2}$$

have negative real part for any $h, \alpha > 0$. Hence, the nonlinear system is locally asymptotically stable. If $\alpha^2 h^2 - 4h < 0$ the equilibrium is a stable spiral point.

(d) Monetary policies based in exogenous monetary expansion or in countercyclical specifications do not stabilize inflation and money demand. However, Obst's Rule does.