TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS – 2020/2021

Sheet 2. Difference Equations (1)

Solutions

2-1. *Classify the following difference equations*
(a) $x_{t+1} = x_t^2 - e^t$;
(b) $x_{t+1} = x_t - e^t$;
(c) $x_{t+1} = 3.2x_t(1 - 0.25x_t)$;
(d) $x_{t+1} - x_t = -\frac{4}{3}x_t$;
(e) $x_{t+1}(2 + 3x_t) = 4x_t$;
(f) $x_{t+2} = 3x_{t+1} - x_t + t$;
(g) $x_{t+4} - x_{t+3} = \sqrt[3]{x_{t+1}}$.

Solution:
(a) first–order, non–linear, non–autonomous;
(b) first–order, linear, non–autonomous;
(c) first–order, non–linear, autonomous;
(d) first–order, linear, autonomous, with constant coefficients;
(e) first–order, non–linear, autonomous;
(f) second–order, linear, non–autonomous;
(g) fourth–order, non–linear, autonomous

2-2. *Check that the following sequences are solution of the corresponding difference equation*
(a) $x_t = 2^t$; $x_{t+2} = x_{t+1} + 2x_t$;
(b) $x_t = \frac{t(t+1)}{2}$; $x_{t+1} = x_t + t + 1$;
(c) $x_t = \cos \pi t$; $x_{t+1} = -x_t$.

Solution:
Note that
(a) $x_{t+1} + 2x_t = 2^{t+1} + 22^t = 22^{t+1} = 2^{t+2} = x_{t+2}$;
(b) $x_t + t + 1 = \frac{t(t+1)}{2} + t + 1 = \frac{t(t+1)+2(t+1)}{2} = \frac{(t+1)(t+2)}{2} = x_{t+1}$.
(c) $-x_t = -\cos \pi t = \cos(\pi t + \pi) = \cos \pi(t+1) = x_{t+1}$.

2-3. *Consider the difference equation $x_{t+1} = \sqrt{x_t - 1}$ with $x_0 = 5$. Compute x_1, x_2. and x_3. What about x_4?*

Solution: $x_1 = \sqrt{5-1} = 2$, $x_3 = \sqrt{2-1} = 1$, $x_4 = \sqrt{1-1} = 0$ and $x_5 = \sqrt{0-1}$ has no sense. This illustrate the importance of taking into consideration the domain of function f that defines the recursion. For the recursion be well defined, it is necessary to find an interval $[a,b]$ such that $f([a,b]) \subset [a,b]$. This is impossible for this difference equation.

2-4. *Find the solutions of the following difference equations with the given values of x_0:*
(a) $x_{t+1} = 2x_t + 4$, $x_0 = 1$;
(b) $x_{t+1} = -0.5x_t + 3$, $x_0 = 1$;
(c) $2x_{t+1} + 3x_t + 2 = 0$, $x_0 = -1$;
(d) $x_{t+1} - x_t = -\frac{4}{3}x_t$, $x_0 = 3$.
Study the long run behavior of the solutions.

Solution: We know that the solution of the equation $x_{t+1} = ax_t + b$, $a, b \in \mathbb{R}$ with initial condition x_0 is given by

$$x_t = a^t(x_0 - x^0) + x^0, \quad \text{where} \quad x^0 = \frac{b}{1-a},$$

when $a \neq 1$, and $x_t = x_0 + tb$ when $a = 1$. We will apply these formulas to solve the difference equations above.
(a) Here $a = 2$ and $b = 4$, thus $x^0 = 4/(1-2) = -4$ and then $x_t = 5\,2^t - 4$ diverges to $+\infty$;
(b) Here $a = -0.5$ and $b = 3$, thus $x^0 = 3/(1+0.5) = 2$ and then $x_t = -(-0.5)^t + 2$ converges to 2;
(c) Here $a = -1.5$ and $b = -1$, thus $x^0 = -1/(1+1.5) - 0.4$ and then $x_t = -0.6(-1.5)^t - 0.4$ oscillates, does not converge;
(d) Here $a = -1/3$ and $b = 0$, thus $x^0 = 0$ and then $x_t = 3(-1/3)^t$ converges to 0.

2-5. *The income Y_t evolves according to the equation*

$$Y_t = C_t + I_t,$$

where I_t denotes investment and C_t is consumption. Supposing that $C_t = mY_t + c$, with $0 \le m < 1$, $c > 0$, and that $I_t = I$ is constant, find a difference equation for income Y_t, solve it, and study the long run behavior of the solution.

Solution: To find a difference equation for income Y_t, substitute C_t into the equation for Y_t to get

$$Y_{t+1} = mY_t + c + I.$$

Given Y_0, the solution is (see problem above for the expression of the solution of a linear first–order difference equation)

$$Y_t = \frac{c+I}{1-m} + m^t \left(Y_0 - \frac{c+I}{1-m} \right).$$

As we can see, Y_t converges to $\dfrac{c+I}{1-m}$ as $t \to \infty$.

2-6. *Let S_0 denotes an initial sum of money. There are two basic methods for computing the interest earned in a period, for example, one year:*

(a) *S_0 earns* simple interest *at rate r if each period the interest equals a fraction r of S_0.*

(b) *S_0 earns* compound interest *at rate r if each period the interest equals a fraction r of the sum accumulated at the beginning of that period.*

Find a difference equation for the two models above, and find the solution.

Solution:

With S_t denoting the amount of capital at period t, we get:

(a) $S_{t+1} = S_t + rS_0$. The solution of this first–order linear equation is

$$S_t = S_0(1 + tr).$$

(b) $S_{t+1} = S_t + rS_t$. This is a geometrical sequence

$$S_t = (1+r)^t S_0.$$

2-7. *Given demand and supply for the cobweb model as follows, find the intertemporal equilibrium price, and determine whether the equilibrium is stable:*

(a) *$Q_d = 18 - 3P$, $Q_s = -3 + 4P$;*

(b) *$Q_d = 22 - 3P$, $Q_s = -2 + P$;*

(c) *$Q_d = 16 - 6P$, $Q_s = 6P - 5$;*

Solution: Recall that if $Q_d = \alpha - \beta P$ and $Q_s = -\gamma + \delta P$, then the solution converges to the equilibrium price $(\alpha + \gamma)/(\beta + \delta)$ iff $\delta/\beta < 1$.

(a) $Q_d = 18 - 3P$, $Q_s = -3 + 4P$. Here $\delta = 4$, $\beta = 3$, $\delta/\beta = 4/3$, thus P_t does not converges.

(b) $Q_d = 22 - 3P$, $Q_s = -2 + P$; Here $\delta = 1$, $\beta = 3$, $\delta/\beta = 1/3$, thus P_t converges to $P^0 = 6$.

(c) $Q_d = 16 - 6P$, $Q_s = 6P - 5$; Here $\delta = 6$, $\beta = 6$, $\delta/\beta = 1$, thus $P_t = P_0$ for every t.

2-8. *In the cobweb model, suppose that the market clearance still holds in each period, $Q_{d,t} = Q_{s,t}$, but that the the supply function is determined not by the price at the previous period, $Q_{s,t} = S(P_{t-1})$, but by the* expected price *at period t:*

$$Q_{s,t} = -\gamma + \delta P_t^*.$$

Sellers form expectations about the price according to the following adaptive rule:

$$P_{t+1}^* = P_t^* + \eta(P_t - P_t^*), \qquad 0 < \eta \le 1,$$

where η is an expectation–adjustment parameter.

(a) *Give an economic interpretation to the preceding equation;*

(b) *Is the cobweb model a particular case of the present model?*

(c) *Find a difference equation for this model;*

(d) *Find the trajectory of price. Is this path necessarily oscillatory? Can it be oscillatory? Under what circumstances?*

(e) *Show that the time path P_t, if oscillatory, will converge only if $1 - 2/\eta < -\delta/\beta$. As compared with the cobweb model without adaptive expectations, does the present model have a wider or narrower range for the stability–inducing values of $-\delta/\beta$?*

Solution:

(a) The equation

$$P^*_{t+1} - P^*_t = \eta(P_t - P^*_t) \tag{1}$$

means that sellers revise their previous expectations of "normal" price in each period in proportion to the difference between actual price and what was previously considered to be "normal".

(b) When $\eta = 1$ it is the cobweb model.

(c) From the equality $Q_{d,t} = \alpha - \beta P_t = -\gamma + \delta P^*_t = Q_{s,t}$ we solve for P^*_t to get

$$P^*_t = \frac{1}{\delta}(\alpha + \gamma - \beta P_t),$$

and then

$$P^*_{t+1} = \frac{1}{\delta}(\alpha + \gamma - \beta P_{t+1}).$$

Substituting P^*_t and P^*_{t+1} into (1) and regrouping terms we find

$$P_{t+1} = \left(1 - \eta - \frac{\eta\delta}{\beta}\right) P_t + \frac{\eta(\alpha+\gamma)}{\beta}.$$

(d) The solution of the difference equation of item above is

$$P_t = \frac{\alpha+\gamma}{\beta+\delta} + \left(1 - \eta - \frac{\eta\delta}{\beta}\right)^t \left(P_0 - \frac{\alpha+\gamma}{\beta+\delta}\right), \qquad t = 0, 1, 2, \ldots.$$

The path is oscillatory iff

$$1 - \eta - \frac{\eta\delta}{\beta} < 0.$$

It is obvious that choosing η small enough, the above quantity is positive, thus the path is not necessarily oscillatory. The path is oscillatory iff

$$\frac{1}{1 + \frac{\delta}{\beta}} < \eta \leq 1. \tag{2}$$

(e) As proved in the item above, the path is oscillatory iff (2) holds. It will converge whenever

$$-1 < 1 - \eta - \frac{\eta\delta}{\beta} < 0,$$

or, rearranging terms, whenever

$$\frac{\delta}{\beta} < \frac{2}{\eta} - 1.$$

In the traditional cobweb model, the condition of stability is $\delta/\beta < 1$. The possibility of stability is much improved when adaptive expectations are assumed. For a numerical example, suppose that $\delta = 2$, $\beta = 1$ and $\eta = 1/4$. While $\delta/\beta = 2$ and thus there is no stability in the traditional cobweb model, we have $\frac{\delta}{\beta} = 2 < 7 = \frac{2}{\eta} - 1$ and in the adaptive model we still have convergence.