TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

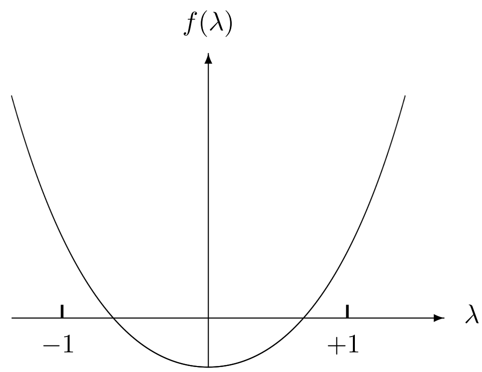
Sheet 4. Difference Equations (3)

Solutions

4-1. *Prove the equivalence of the two following assertions:*
 (a) *The quadratic equation* $\lambda^2 - p\lambda + q = 0$ *has roots satisfying* $|\lambda| < 1$*;*
 (b) $|p| < 1 + q$ *and* $q < 1$ *(*Jury condition*).*

Solution: The function $f(\lambda) = \lambda^2 - p\lambda + q$ is a convex parabola. Let us distinguish two different cases.

- Complex roots. This happens when $q > p^2/4$ (so $q > 0$). The roots are
$$\lambda_{1,2} = \frac{1}{2}\Big(p \pm i\sqrt{4q - p^2}\Big),$$
with modulus $|\lambda_{1,2}| = \frac{1}{2}\sqrt{p^2 + 4q - p^2} = \sqrt{q}$. Hence (b) implies (a). Now, to prove the reverse, suppose that (a) holds. Then, $q < 1$. On the other hand, as $f(\lambda) > 0$ for all λ, $f(1) = 1 - p + q > 0$ and $f(-1) = 1 + p + q > 0$. These inequalities are equivalent to $|p| < 1 + q$, hence (a) implies (b).
- Real roots. This happens when $q \le p^2/4$. Note in the drawing below that $\lambda_{1,2} \in (-1, 1)$ if and only if $f(1) = 1 - p + q > 0$, $f(-1) = 1 + p + q > 0$, $f'(-1) = 2 - p > 0$ and $f'(1) = 2 + p < 0$. The first two inequalities give $|p| < 1 + q$ and the second ones $|p| < 2$. They are equivalent to $|p| < 1 + q$ and $q < 1$.



4-2. *Find the interval of values of the parameter a for which the system*
$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & -\frac{1}{2} \\ 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$
is globally asymptotically stable. Find the stable manifold (if possible) for the cases $a = -5/3$ and $a = 5/2$.

Solution: We will apply the Jury condition proved in the problem above. The characteristic polynomial of the matrix of the system is
$$p_A(\lambda) = \lambda^2 - (a + \frac{1}{2})\lambda + 1 + \frac{a}{2}.$$
Here, $p = a + \frac{1}{2}$ and $q = 1 + \frac{a}{2}$. The roots are smaller than 1 in module iff $|p| < 1 + q$ and $q < 1$. The second inequality gives
$$1 + \frac{a}{2} < 1 \Rightarrow a < 0.$$
The first inequality can be decomposed into
$$a + \frac{1}{2} < 2 + \frac{a}{2} \Rightarrow a < 3$$
and
$$-\left(a + \frac{1}{2}\right) < 2 + \frac{a}{2} \Rightarrow a > -\frac{5}{3}.$$
Thus, the system is g.a.s. iff $-\frac{5}{3} < a < 0$.

To answer the other questions, suppose first that $a = -5/3$. In this case the eigenvalues are -1 and $-1/6$. The stable manifold is the eigenspace $S(-1/6)$. It is easy to find

$$S(-1/6) = \{(x, y) \,:\, 3x + y = 0\}.$$

Thus, trajectories with initial conditions $3x_0 + y_0 = 0$ converge to the equilibrium point $(0, 0)$. When $a = 5/2$ there is a double eigenvalue, $\lambda = 3/2 > 1$, thus there is no stable manifold.

4-3. *Suppose that a firm starts activity with 100 machines of the same age. After 2 years, machines become obsolete and must be replaced for a new one. Moreover, it is known that 11% of the machines of 1 year will fail and must be also replaced. Write down the equations of the dynamical system involved and the initial condition.*

Solution: Denote by x_t the percentage of machines of age 0 at time t, y_t the percentage of machines of age 1 at time t and z_t the percentage of machines of age 2 at time t. The dynamics is

$$\begin{aligned} x_{t+1} &= z_t + 0.11 y_t, \\ y_{t+1} &= x_t, \\ z_{t+1} &= y_t - 0.11 y_t = 0.89 y_t. \end{aligned}$$

The matrix of the system is

$$\begin{pmatrix} 0 & 0.11 & 1 \\ 1 & 0 & 0 \\ 0 & 0.89 & 0 \end{pmatrix}.$$

4-4. *Consider the following deterministic version of a model of inequality transmission across generations of Gary Solon*[1]*. Parents invest I_{t-1} into the child's human capital h_t, with effect $h_t = \theta \ln I_{t-1} + g_t$, where $\theta > 0$ and g_t is human endowment the child receives independently of parent's investment (genetic inheritance). Assume that $g_t = \delta + \lambda g_{t-1}$, $0 < \lambda < 1$. Lifetime income of the child is given by $\ln y_t = \mu + p h_t$.*

(a) *Interpret the coefficients θ, λ and p.*
(b) *Assuming that parents' investment is a positive constant fraction of their income, $I_{t-1} = e^k y_{t-1}$, $k \leq 0$, write down a second order difference equation for $\ln y_t$.*
(c) *Show that in this model the log of income of the child is positively related with the parents' log of income, but inversely related with the grandfather's log of income.*
(d) *Show that the model never shows an oscillating behavior.*
(e) *Find the general solution when $\lambda = \gamma$ and show in this case that the income converges to an equilibrium.*

Solution:

(a) θ marginal product for human capital investment; λ heritability coefficient; p earnings return to human capital.
(b) Let $z_t = \ln y_t$. After plugging both human capital and the investment into the equation for $\ln y_t$ we get

$$z_t = \mu^* + \gamma z_{t-1} + p g_t,$$

where $\mu^* = \mu + \gamma k$ and $\gamma = \theta p$. Lagging the equation one period we get

$$z_{t-1} = \mu^* + \gamma z_{t-2} + p g_{t-1}.$$

Multiplying this equation by λ and subtracting to the former equation we obtain

$$z_t = \mu^*(1 - \lambda) + p\delta + (\gamma + \lambda) z_{t-1} - \gamma\lambda z_{t-2}.$$

(c) $\gamma + \lambda > 0$ and $-\gamma\lambda < 0$.
(d) The characteristic equation is

$$r^2 - (\gamma + \lambda) r + \gamma\lambda = 0.$$

The discriminant is $(\gamma + \lambda)^2 - 4\gamma\lambda = (\lambda - \gamma)^2 \geq 0$.
(e) $z_t = C_1 \lambda^t + C_2 t \lambda^t + z^*$, where $z^* = \frac{\mu^*(1-\lambda) + p\delta}{(1-\lambda)^2}$.

[1] Gary Solon. "Theoretical models of inequality transmission across multiple generations". (18790), February 2013.

4-5. *K is a student with the following habit: once she studies one day, it is likely that she will not study the following day with probability* 0.7. *On the other hand, the probability that she does not study two consecutive days is* 0.6. *Assuming that today K has promised to study, with which probability does K study in the long run?*

Solution: Let us denote E_t (N_t) the event of studying (not studying) at day t, and by e_t (n_t) the probability of E_t (N_t). Note that $e_t + n_t = 1$ for all t. We have the following data

$$P(N_{t+1}|E_t) = 0.7, \quad P(N_{t+1}|N_t) = 0.6,$$

where $P(A|B)$ means the probability of event A conditioned by event B. Hence $P(E_{t+1}|E_t) = 0.3$ and $P(E_{t+1}|N_t) = 0.4$. Since

$$P(E_{t+1}) = P(E_{t+1}|E_t)P(E_t) + P(E_{t+1}|N_t)P(N_t)$$

we get the difference equation

$$e_{t+1} = 0.3e_t + 0.4n_t.$$

Analogously, we have

$$n_{t+1} = 0.7e_t + 0.6n_t.$$

The eigenvalues of the matrix system $A = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{pmatrix}$ are $\lambda_1 = 1$ and $\lambda_2 = -0.1$. The matrix is diagonalizable, with $D = \begin{pmatrix} 1 & 0 \\ 0 & -0.1 \end{pmatrix}$. It is easy to find that a diagonalization matrix is

$$P = \begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{11}\begin{pmatrix} 1 & 1 \\ 7 & -4 \end{pmatrix}.$$

Hence, recalling the formula for the solution

$$\begin{pmatrix} e_t \\ n_t \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 4 & 1 \\ 7 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda_2^t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 7 & -4 \end{pmatrix}\begin{pmatrix} e_0 \\ n_0 \end{pmatrix},$$

where e_0 (n_0) is the probability that K. studies (not study) today. One finds

$$e_t = \frac{1}{11}\lambda_2^t(7e_0 - 4n_0) + \frac{4}{11}(e_0 + n_0) \quad \stackrel{t\to\infty}{\longrightarrow} \quad \frac{4}{11}(e_0 + n_0).$$

Thus, (e_t, n_t) converges to the stationary distribution $(4/11, 7/11)$ independently of the values of e_0 and n_0, that means that K. will study in the long run with probability 0.3636.

Remark. 1) Note that as $n_t = 1 - e_t$, we can reduce the system to the single equation

$$e_{t+1} = -0.1e_t + 0.4.$$

The equilibrium point is of course $e^* = 4/11$ and the solution converges to e^* since the coefficient of e_t is smaller than 1 in absolute value. The solution is

$$e_t = e^* + (-0.1)^t(e^* - e_0).$$

2) The matrix $A - I$ is not regular. This is the reason why the homogeneous system above has equilibrium points different from $(0, 0)$. Note that this case has not been studied in the theory notes. However, the problem can be solved because we know the solution.

4-6. *A psychologist places a mouse inside a jail with two doors, A and B. Going through door A, the mouse receive an electrical shock. The mouse never chooses door A twice in a row. Some food is behind door B. After choosing B, the probability of returning to B in the following day is* 0.6. *At the beginning of the experiment (Monday), the mouse chooses A or B with the same probability.*
(a) *With which probability does the mouse choose door A on Thursday?*
(b) *Which is the stationary distribution of this experiment?*
(c) *What do you think the mouse thinks about the psychologist?*

Solution: Denote by a_t (b_t) the probability of choosing door A (B) in day t. We suppose that $a_t + b_t = 1$ for every t, that is, the mouse cannot go out of the jail. The data of the problem says that $a_0 = b_0 = 1/2$ (thus, $t = 0$ corresponds to the first Monday). Then, the probabilities must satisfy (see problem above for details)

$$a_{t+1} = 0a_t + (1 - 0.6)b_t,$$
$$b_{t+1} = a_t + 0.6b_t.$$

The matrix system has eigenvalues $\lambda = 1$ and $\lambda = -0.4$. It is easy to show that a diagonalization matrix is given by

$$P = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}.$$

The solution is then

$$\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-0.4)^t \end{pmatrix}\frac{1}{7}\begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix},$$

from which

$$a_t = \frac{2}{7} + \frac{3}{14}(-0.4)^t, \qquad t \geq 0.$$

(a) Thursday corresponds with $t = 3$, thus the probability of choosing door A in Thursday is

$$a_3 = 0.272.$$

(b) There exists an stationary distribution, given by the limit

$$\lim_{t\to\infty} a_t = \frac{2}{7} \approx 0.2857.$$

The stationary distribution is to choose door A with probability $2/7$ and door B with probability $5/7$.

(c) It's a joke.

4-7. ***Phillips curve I.***
The Phillips curve *relates negatively the rate of growth of money wage w and the unemployment rate U,*

(1) $$w = f(U), \qquad f'(U) < 0.$$

This was justified empirically by A.W. Phillips for the U.K. in a very influential paper[2]*. Later, the relation was postulated to affect also to the rate of inflation, p, since a growing–money wage costs would had inflationary effects*[3]*,*

$$p = w - T.$$

Here, T denotes an exogenous increase in labor productivity (hence inflation appears only if the salary grows faster than productivity). Assuming a linear form of function f, $f(U) = \alpha - \beta U$, we will have that at every $t \geq 1$

(2) $$p_t = \alpha - T - \beta U_t, \qquad \alpha, \beta > 0.$$

On the other hand, the theory links the unemployment rate and the rate of inflation according to

(3) $$U_{t+1} - U_t = -k(m - p_t), \qquad 0 < k \leq 1,$$

where m is the rate of growth of the nominal money balance[4]*. Noticing that $m - p$ is the rate of growth of real money, Eqn.* (3) *establishes that the rate of growth of unemployment is negatively related with the rate of growth of real money.*

Find a difference equation for U_t and study the stability properties of the solution.

Solution: Substituting the Phillips relation into the equation satisfied by the unemployment rate we get

$$U_{t+1} - U_t = -k(m - \alpha + T + \beta U_t)$$

and rearranging terms

$$U_{t+1} = (1 - k\beta)U_t - k(m - \alpha + T).$$

The particular solution U^*, that is also the equilibrium point of the equation is given by

$$U^* = (1 - k\beta)U^* - k(m - \alpha + T) \Rightarrow U^* = \frac{\alpha - m - T}{\beta}.$$

As we know, the equation is g.a.s. and converges to U^* as $t \to \infty$ iff

$$|1 - k\beta| < 1 \Leftrightarrow 0 < k\beta < 2.$$

Moreover, observe that when $0 < k\beta < 1$, the solution converges monotonically to U^*; when $1 < k\beta < 2$ converges to U^* in a oscillating fashion, since the coefficient of U_t in the difference equation for U is $1 - k\beta < 0$, and when $k\beta = 1$ we have an uninteresting case.

[2]A.W. Phillips (1956) "The relationship between unemployment and the rate of change of money wage rates in the United Kingdom," *Economica*, November 1958, pp. 283–299.

[3]The rate of growth of money wage is $(W_{t+1} - W_t)/W_t$, where W_t is wage at time t; the rate of inflation is the rate of the general price level, $p = (P_{t+1} - P_t)/P_t$.

[4]That is, $m = (M_{t+1} - M_t)/M_t$, where M_t is the nominal money balance, fixed by the monetary authority. It is supposed here that m constant, independent of t.

4-8. ***Phillips curve II.***
Continuing with the Phillips' model, we analyze now the modification introduced by Friedman[5], *considering the* expected–augmented *version of the Phillips relation*

(4) $$w = f(U) + g\pi, \qquad (0 < g \leq 1),$$

where π denotes the expected rate of inflation. The idea is that if an inflationary trend has been observed long enough, people form certain inflation expectations, which they attempt to incorporate into their money–wage demands. Then, (2) *results in the equation*

(5) $$p_t = \alpha - T - \beta U_t + g\pi_t, \quad t \geq 0.$$

How is formed inflation expectations? Commonly is is assumed the adaptive expectations *hypothesis*

(6) $$\pi_{t+1} - \pi_t = j(p_t - \pi_t), \qquad 0 < j \leq 1.$$

This means that when the actual rate of inflation p turns out to exceed the expected rate π, the latter, having now been proven to be too low, is revised upward. Conversely, if p falls short of π, then π is revised in the downward direction. The speed of adjustment is j.

Consider the model given by Eqs. (3), (5) *and* (6).

(a) *Eliminate p_t and write a system of linear difference equations for the variables (U_t, π_t).*
(b) *Using the Jury condition, determine whether the system is g.a.s.*
(c) *Find and interpret the fixed or equilibrium points of the system.*

Solution:

(a) Substituting p_t into (3) and (6) we have

$$\begin{pmatrix} U_{t+1} \\ \pi_{t+1} \end{pmatrix} = \overbrace{\begin{pmatrix} 1-k\beta & kg \\ 1-j(1-g) & -\beta j \end{pmatrix}}^{A} \begin{pmatrix} U_t \\ \pi_t \end{pmatrix} + \begin{pmatrix} k(\alpha - m - T) \\ j(\alpha - T) \end{pmatrix}$$

(b) The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - \overbrace{(2 - j(1-g) - k\beta)}^{p}\,\lambda + \overbrace{(1 - j(1-g) - k\beta(1-j))}^{q}\,.$$

Recall that the Jury condition assuring that the module of the roots are smaller than 1 is $|p| < 1 + q < 2$. Applied to our model obtains

$$|2 - j(1-g) - k\beta| < 2 - j(1-g) - k\beta(1-j) < 2.$$

Note that the second inequality is always true because all the parameters are positive and $0 < g, j \leq 1$. Thus, the condition reduces to

(7) $$|2 - j(1-g) - k\beta| < 2 - j(1-g) - k\beta(1-j).$$

We have two cases to consider.

(i) $2 - j(1-g) - k\beta \geq 0$. Then, the Jury condition is obviously fulfilled.
(ii) $2 - j(1-g) - k\beta < 0$. Then, Eqn. (7) is

$$k\beta(2-j) + 2j(1-g) < 4.$$

Resuming, the system is g.a.s. iff

$$j(1-g) + k\beta \leq 2$$

or

$$j(1-g) + k\beta > 2 \quad \text{and} \quad k\beta(2-j) + 2j(1-g) < 4.$$

To put an example, suppose that $j = g = 0.5$ and consider the product $k\beta$ as the parameter. Then the above two conditions show that the system is g.a.s. iff $k\beta \in (0, \frac{7}{3})$.

(c) The fixed point of the system (U^*, π^*) is easily found without resorting to the inverse matrix $(I_2 - A)^{-1}$ as follows. It is clear that in any equilibirum solution, the inflation rate p must be also constant, p^* say. After plugging $U_t = U^*$ into (3) we find

$$0 = -k(m - p^*).$$

Thus, $p^* = m$. The equilibrium rate of inflation is equal to the rate of monetary expansion. In the same way, putting $\pi_t = \pi^*$ into (6)

$$0 = j(p^* - \pi^*).$$

[5]M. Friedman (1968) "The role of monetary policy," *American Economic Review*, pp. 1–17.

Thus, $\pi^* = p^* = m$, the expected inflation rate is exactly equal to the actual inflation rate. Finally, substituting $U_t = U^*$ into (5) we get

$$p^* = \alpha - T - \beta U^* + gp^*$$

or

$$U^* = \frac{1}{\beta}(\alpha - T - (1-g)p^*).$$

This is called the *long–run* Phillips relation. As $g \leq 1$, it shows a downward sloping relation between rate of unemployment and inflation rate. When $g = 1$, U^* is independent of p^*. The value of U^* in this case is referred as the *natural rate of unemployment*, which is consistent with any equilibrium rate of inflation.