Topic 2: Difference Equations (II)

4. Second order linear difference equations

Consider the following example.

Example 4.1 (A Multiplier–Accelerator Growth Model)**.** Let Y_t denote national income, C_t total consumption, and I_t total investment in a country at time t. Assume that for $t = 0, 1, \dots,$

(i) $Y_t = C_t + I_t$ (income is divided between consumption and investment)

(ii) $C_{t+1} = aY_t + b$ (consumption is a linear function of previous income)

(iii) $I_{t+1} = c(C_{t+1} - C_t)$ (investment is proportional to to the change in consumption),

where $a, b, c > 0$. Find a second order difference equation describing this national economy.

Solution: We eliminate two of the unknown functions as follows. From (i), we get (iv) $Y_{t+2} = C_{t+2} + I_{t+2}$. Replacing I_{t+2} from (iii) we get $Y_{t+2} = C_{t+2} + c(C_{t+2} - C_{t+1}) = (1 + c)C_{t+2} - cC_{t+1}$. Finally, use (ii) to obtain

$$Y_{t+2} - a(1 + c)Y_{t+1} + acY_t = b, \qquad t = 0, 1, \dots$$

The form of the solution depends on the coefficients a, b, c.

The second–order linear difference equation is

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = b_t,$$

where a_0 and $a_1$are constants and b_t is a given function of t. The associated *homogeneous equation* is

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = 0,$$

and the associated *characteristic equation* is

$$r^2 + a_1 r + a_0 = 0.$$

This quadratic equation has solutions

$$r_1 = -\frac{1}{2}a_1 + \frac{1}{2}\sqrt{a_1^2 - 4a_0}, \qquad r_2 = -\frac{1}{2}a_1 - \frac{1}{2}\sqrt{a_1^2 - 4a_0}.$$

When the sign of the discriminant, $a_1^2 - 4a_0$, is negative, the solutions are (conjugate) complex numbers. Recall that a complex number is $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$ is called the imaginary unit, so that $i^2 = -1$. The *real part* of z is a, and the *imaginary part* of z is b. The conjugate of $z = a + ib$ is $\overline{z} = a - ib$. Complex numbers can be added, $z + z' = (a + a') + i(b + b')$, and multiplied,

$$zz' = (a + ib)(a' + ib') = aa' + iab' + ia'b + i^2 bb' = (aa' - bb') + i(ab' + a'b).$$

For the following theorem we need the *modulus of* z, $\rho = |z| = \sqrt{a^2 + b^2}$, and the angle $\theta \in [-\pi/2, \pi/2]$ such that $\tan\theta = b/a$. It is useful to recall the following table of trigonometric values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

For the negative values of the argument θ, use the properties of sine and cosine: $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$.

For example, the modulus and argument of $1 - i$ is $\rho = \sqrt{2}$ and $\theta = -\pi/4$, respectively, since $\tan\theta = -1/1 = -1$. For $z = -\sqrt{3} - i3$, the module is $\rho = \sqrt{12}$ and the argument is $\tan\theta = \sqrt{3}$, hence $\theta = \pi/3$.

Theorem 4.2. *The general solution of*

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = 0 \quad (a_0 \neq 0) \tag{4.1}$$

is as follows:

(1) If the characteristic equation has two distinct real roots,

$$x_t = A r_1^t + B r_2^t.$$

(2) If the characteristic equation has one real double root,

$$x_t = (A + Bt) r^t.$$

(3) If the characteristic equation has complex roots $r_{1,2} = a \pm ib$

$$x_t = \rho^t (A\cos\theta t + B\sin\theta t),$$

where $\rho = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$, $\theta \in [-\pi/2, \pi/2]$.

A and B are arbitrary constants.

Remark 4.3. When the characteristic equation has complex roots, the solution of (4.1) involves oscillations. Note that when $\rho < 1$, ρ^t tends to 0 as $t \to \infty$ and the oscillations are damped. If $\rho > 1$, the oscillations are explosive, and in the case $\rho = 1$, we have undamped oscillations.

Example 4.4. Find the general solutions of

(a) $x_{t+2} - 7x_{t+1} + 6x_t = 0$, (b) $x_{t+2} - 6x_{t+1} + 9x_t = 0$, (c) $x_{t+2} - 2x_{t+1} + 4x_t = 0$.

Solution: (a) The characteristic equation is $r^2 - 7r + 6 = 0$, whose roots are $r_1 = 6$ and $r_2 = 1$, so the general solution is

$$x_t = A6^t + B, \qquad A, B \in \mathbb{R}.$$

(b) The characteristic equation is $r^2 - 6r + 9 = 0$, which has a double root $r = 3$. The general solution is

$$x_t = 3^t(A + Bt).$$

(c) The characteristic equation is $r^2 - 2r + 4 = 0$, with complex solutions $r_1 = \frac{1}{2}(2 + \sqrt{-12}) = (1 + i\sqrt{3})$, $r_2 = (1 - i\sqrt{3})$. Here $\rho = 2$ and $\tan\theta = -\frac{\sqrt{12}}{-2} = \sqrt{3}$. this means that $\theta = \pi/3$. The general solution is

$$x_t = 2^t \left(A\cos\frac{\pi}{3}t + B\sin\frac{\pi}{3}t \right).$$

4.1. **The nonhomogeneous equation.** Now consider the nonhomogeneous equation

$$x_{t+2} + a_1 x_{t+1} + a_0 x_t = b_t, \tag{4.2}$$

and let x_t^* be a *particular solution*. It turns out that solutions of the equation have an interesting structure, due to the linearity of the equation.

Theorem 4.5. *The general solution of the nonhomogeneous equation* (4.2) *is the sum of the general solution of the homogeneous equation* (4.1) *and a particular solution* x_t^* *of the nonhomogeneous equation.*

Example 4.6. Find the general solution of $x_{t+2} - 4x_t = 3$.

Solution: Note that $x_t^* = -1$ is a particular solution. To find the general solution of the homogeneous equation, consider the solutions of the characteristic equation, $m^2 - 4 = 0$, $m_{1,2} = \pm 2$. Hence, the general solution of the nonhomogeneous equation is

$$x_t = A(-2)^t + B2^t - 1.$$

Example 4.7. Find the general solution of $x_{t+2} - 4x_t = t$.

Solution: Now it is not obvious how to find a particular solution. We can try with the *method of undetermined coefficients* and try with some expression of the form $x_t^* = Ct + D$. Then, we look for constants a, b such that x_t^* is a solution. This requires

$$C(t+2) + D - 4(Ct + D) = t, \qquad \forall t = 0, 1, 2, \ldots$$

One must have $C - 4C = 1$ and $2C + D - 4D = 0$. It follows that $C = -1/3$ and $D = -2/9$. Thus, the general solution is

$$x_t = A(-2)^t + B2^t - t/3 - 2/9.$$

Example 4.8. Find the solution of $x_{t+2} - 4x_t = t$ satisfying $x_0 = 0$ and $x_1 = 1/3$.

Solution: Using the general solution found above, we have two equations for the two unknown parameters A and B:

$$\left.\begin{array}{ll} A + B + \frac{2}{9} & = 0 \\ -2A + 2B - \frac{1}{3} + \frac{2}{9} & = \frac{1}{3} \end{array}\right\}.$$

The solution is $A = -2/9$ and $B = 0$. Thus, the solution of the nonhomogeneous equation is

$$x_t = -\frac{2}{9}(-2)^t - \frac{t}{2} + \frac{2}{9}.$$

Remark 4.9. The method of undetermined coefficients for solving equation (4.2) suppose that a particular solution has the form of the nonhomogeneous term, b_t. The method works quite well when this term is of the form

$$b^t, \quad t^m, \quad \cos bt, \quad \sin bt$$

or linear combinations of them. The concrete form of the particular solution depends also on the roots of the characteristic polynomial. We give the following directions.

(1) If $b_t = ab^t$, where b is not a root of the characteristic polynomial, then we try $x_t^* = Cb^t$.
(2) If $b_t = ab^t$, where b is a root of the characteristic polynomial of multiplicity p, then we try with $x_t^* = Ct^p b^t$.
(3) If $b_t = b_m t^m + b_{m-1}t^{m-1} + \cdots + b_0$ is a polynomial of degree m, we distinguish two cases:
   (a) If 1 is not a root of the characteristic polynomial, then we try with $x_t^* = C_m t^m + C_{m-1}t^{m-1} + \cdots + C_0$.
   (b) If 1 is a root of the characteristic polynomial of multiplicity p, then we try with $x_t^* = t^p(C_m t^m + C_{m-1}t^{m-1} + \cdots + C_0)$.

Finally, substitute x_t^* into the equation and determine the coefficients C's.

Example 4.10. Solve the equation $x_{t+2} - 5x_{t+1} + 6x_t = 4^t + t^2 + 3$.

Solution: The homogeneous equation has characteristic equation $r^2 - 5r + 6 = 0$, with two different real roots $r_{1,2} = 2, 3$. Its general solution is, therefore, $A2^t + B3^t$. To find a particular solution we look for constants C, D, E and F such that a particular solution is

$$x_t^* = C4^t + Dt^2 + Et + F.$$

Plugging this into the equation we find

$$\begin{aligned} C4^{t+2} + D(t+2)^2 + E(t+2) + F - 5(C4^{t+1} + D(t+1)^2 + E(t+1) + F)& \\ + 6(C4^t + Dt^2 + Et + F) = 4^t + t^2 + 3.& \end{aligned}$$

Expanding and rearranging yields

$$2C4^t + 2Dt^2 + (-6D + 2E)t + (-D - 3E + 2F) = 4^t + t^2 + 3.$$

This must hold for every $t = 0, 1, 2, \ldots$ thus,

$$\begin{aligned} 2C &= 1, \\ 2D &= 1, \\ -6D + 2E &= 0, \\ -D - 3E + 2F &= 3. \end{aligned}$$

It follows that $C = 1/2$, $D = 1/2$, $E = 3/2$ and $F = 4$. The general solution is

$$x_t = A2^t + B3^t + \frac{1}{2}4^t + \frac{1}{2}t^2 + \frac{3}{2}t + 4.$$

Example 4.11. Solve $x_{t+2} - 5x_{t+1} + 4x_t = 4^t + t^2 + 3$.

Solution: The homogeneous equation has characteristic equation $r^2 - 5r + 4 = 0$, with two different real roots, $r_{1,2} = 1, 4$. Its general solution is, therefore, $A + B4^t$. To find a particular solution we look for constants C, D, E y F such that a particular solution is

$$x_t^* = Ct4^t + t(Dt^2 + Et + F).$$

Plugging this into the equation we find

$$\begin{aligned} &C(t+2)4^{t+2} + (t+2)(D(t+2)^2 + E(t+2) + F) - 5C(t+1)4^{t+1} \\ &\qquad - 5(t+1)(D(t+1)^2 + E(t+1) + F) + 4Ct4^t + 4t(Dt^2 + Et + F) = 4^t + t^2 + 3. \end{aligned}$$

Equating coefficients we find

$$\begin{aligned} t4^t :&\quad 16C - 20C + 4C = 0, \\ 4^t :&\quad 32C - 20C = 1 \Rightarrow C = \frac{1}{12}, \\ t^3 :&\quad D - 5D + 4D = 0, \\ t^2 :&\quad 8D + E - 10D - 5E + 4E = 1 \Rightarrow D = -\frac{1}{2}, \\ t :&\quad 4D + 8D + 4E + F - 10D - 5D - 10E - 5F + 4F = 0 \Rightarrow E = \frac{1}{4}, \\ t^0 :&\quad 8D + 4F + 2F - 5D - 5E - 5F = 3 \Rightarrow F = \frac{23}{4}. \end{aligned}$$

5. Linear systems of difference equations

Now we suppose that the dynamic variables are vectors, $X_t \in \mathbb{R}^n$. A *first order system of linear difference equations with constant coefficients* is given by

$$\begin{aligned} x_{1,t+1} &= a_{11}x_{1,t} + \cdots + a_{1n}x_{n,t} + b_{1,t} \\ &\vdots \\ x_{n,t+1} &= a_{n1}x_{1,t} + \cdots + a_{nn}x_{n,t} + b_{n,t} \end{aligned}$$

An example is

$$\begin{aligned} x_{1,t+1} &= 2x_{1,t} - x_{2,t} + 1 \\ x_{2,t+1} &= x_{1,t} + x_{2,t} + e^{-t}. \end{aligned}$$

Most often we will rewrite systems omitting subscripts using different letters for different variables, as in

$$\begin{aligned} x_{t+1} &= 2x_t - y_t + 1 \\ y_{t+1} &= x_t + y_t + e^{-t}. \end{aligned}$$

A linear system is equivalent to the matrix equation

$$X_{t+1} = AX_t + B_t,$$

where

$$X_t = \begin{pmatrix} x_{1,t} \\ \vdots \\ x_{n,t} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{1,t} \\ \vdots \\ b_{n,t} \end{pmatrix}$$

We will center on the case where the independent term $B_t \equiv B$ is a constant vector.

5.1. **Homogeneous systems.** Consider the homogeneous system $X_{t+1} = AX_t$.

Note that $X_1 = AX_0$, $X_2 = AX_1 = AAX_0 = A^2X_0$. Thus, given the initial vector X_0, the solution is

$$X_t = A^t X_0, \quad t = 0, 1, \dots$$

5.2. **Non Homogeneous Systems.** Consider the non-homogeneous system

$$X_{t+1} = AX_t + B, \tag{5.1}$$

where B is a constant vector. Assume that there is only one equilibrium point, X^0. This si true if $|A - I| \neq 0$, because then

$$X^0 = (I - A)^{-1} B.$$

Let $Y_t = X_t - X^0$ and rewrite the system in terms of Y_t

$$\begin{aligned} Y_{t+1} &= X_{t+1} - X^0 = AX_t + B - X^0 = AY_t + AX^0 + B - X^0 \\ &= AY_t + B - (I - A)X^0 = AY_t + B - (I - A)(I - A)^{-1}B \\ &= AY_t, \end{aligned}$$

thus we get an homogeneous system and the solution is thus $Y_t = A^tY_0$ or

$$X_t = X^0 + A^t(X_0 - X^0). \tag{5.2}$$

We have proved the following result.

Theorem 5.1. *Suppose that $|A - I_n| \neq 0$. Then the solution of the non-homogeneous system is given by* (5.2)*.*

A question arises immediately. To what extent is this result useful? All depends on how difficult is to compute A^t. When A is diagonalizable, $A^t = PD^tP^{-1}$ for some regular matrix P formed with eigenvectors of A and where D is a diagonal matrix with eigenvalus in its diagonal, so D^t is easily computed.

Example 5.2. Find the general solution of the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

Solution: The matrix $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ has characteristic polynomial $p_A(\lambda) = \lambda^2 - 5\lambda + 6$, with roots $\lambda_1 = 3$ and $\lambda_2 = 2$. Thus, the matrix is diagonalizable. It is easy to find the eigenspaces

$$S(3) =< (1, 1) >, \qquad S(2) =< (1, 2) > .$$

Hence, the matrix P is

$$P=\begin{pmatrix}1&1\\1&2\end{pmatrix},\quad P^{-1}=\begin{pmatrix}2&-1\\-1&1\end{pmatrix},\quad D=\begin{pmatrix}3&0\\0&2\end{pmatrix}$$

and the solution

$$X_t=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}3^t&0\\0&2^t\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}X_0=\begin{pmatrix}2\,3^t-2^t&-3^t+2^t\\2\,3^t-2^{t+1}&-3^t+2^{t+1}\end{pmatrix}\begin{pmatrix}x_0\\y_0\end{pmatrix}$$

Supposing that the initial condition is $(x_0,y_0)=(1,2)$, the solution is given by

$$x_t=2\,3^t-2^t+2(-3^t+2^t)=2^t,$$
$$y_t=2\,3^t-2^{t+1}+2(-3^t+2^{t+1})=2^{t+1}.$$

We give now an alternative method to compute the solution, assuming as before that the matrix A is diagonalizable, with eigenvalues $\lambda_1,\dots,\lambda_n$ (possibly repeated) and eigenvectors $u_1,\dots,u_n$. Let $C_1,\dots,C_n$ be arbitrary constants. Then

$$X_t^h=C_1\lambda_1^tu_1+\cdots+C_n\lambda_n^tu_n$$

is the general solution of the homogeneous system. This is easy to check by the very definition of eigen- values and vectors

$$AX_t^h=C_1\lambda_1^tAu_1+\cdots+C_n\lambda_n^tAu_n=C_1\lambda_1^{t+1}u_1+\cdots+C_n\lambda_n^{t+1}u_n=X_{t+1}^h.$$

The general solution of the complete system is $X_t=X_t^h+X^0$.

Example 5.3. In the above example we found that A was diagonalizable, etc. The general solution is thus ($X^0=0$)

$$X_t=C_13^t\begin{pmatrix}1\\1\end{pmatrix}+C_22^t\begin{pmatrix}1\\2\end{pmatrix}.$$

If we want to find the solution satisfying $X_0=(1,2)$, then e have to determine suitable constants C_1 and C_2 satisfying

$$\begin{pmatrix}1\\2\end{pmatrix}=C_1\begin{pmatrix}1\\1\end{pmatrix}+C_2\begin{pmatrix}1\\2\end{pmatrix}.$$

Hence $C_1=0$ y $C_2=1$.

Example 5.4. Find the general solution of the system

$$\begin{pmatrix}x_{t+1}\\y_{t+1}\end{pmatrix}=\begin{pmatrix}4&-1\\2&1\end{pmatrix}\begin{pmatrix}x_t\\y_t\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}$$

Solution: The fixed point X^* is given by

$$(I_3-A)^{-1}B=\begin{pmatrix}-3&1\\-2&0\end{pmatrix}^{-1}\begin{pmatrix}1\\-1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&-1\\2&-3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}1/2\\5/2\end{pmatrix}$$

We could also have found the point without calculating the inverse and solving instead the system

$$\begin{aligned} x &= 4x - y + 1 \\ y &= 2x + y - 1. \end{aligned}$$

By the example above we already know the general solution of the homogeneous system. The general solution of the nonhomogeneous system is then

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2\,3^t - 2^t & -3^t + 2^t \\ 2\,3^t - 2^{t+1} - 1 & -3^t + 2^{t+1} \end{pmatrix} \begin{pmatrix} x_0 - 1/2 \\ y_0 - 5/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 5/2 \end{pmatrix}.$$

5.3. **Stability of linear systems.** We study here the stability properties of a first order system $X_{t+1} = AX_t + B$ where $|I - A| \neq 0$.

For the following theorem, recall that for a complex number $z = a + bi$, the modulus is $|z| = \rho = \sqrt{a^2 + b^2}$.

Theorem 5.5. *A necessary and sufficient condition for system $X_{t+1} = AX_t + B$ to be g.a.s. is that all roots of the characteristic polynomial $p_A(\lambda)$ (real or complex) have moduli less than 1. In this case, any trajectory converges to $X^0 = (I_n - A)^{-1}B$ as $t \to \infty$.*

We can give an idea of the proof of the above theorem in the case where the matrix A is diagonalizable. As we have shown above, the solution of the nonhomogeneous system in this case is

$$X_t = X^0 + PD^tP^{-1}(X_0 - X^0),$$

where

$$D = \begin{pmatrix} \lambda_1^t & 0 & \dots & 0 \\ 0 & \lambda_2^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^t \end{pmatrix},$$

and $\lambda_1, \dots, \lambda_n$ are the real roots (possibly repeated) of $p_A(\lambda)$. Since $|\lambda_j| < 1$ for all j, the diagonal elements of D^t tends to 0 as t goes to ∞, since $\lambda_j^t \le |\lambda_j|^t \to 0$. Hence

$$\lim_{t\to\infty} X_t = X^0.$$

Example 5.6. Study the stability of the system

$$\begin{aligned} x_{t+1} &= x_t - \frac{1}{2}y_t + 1, \\ y_{t+1} &= x_t - 1. \end{aligned}$$

Solution: The matrix of the system is $\begin{pmatrix} 1 & -1/2 \\ 1 & 0 \end{pmatrix}$, with characteristic equation $\lambda^2 - \lambda + 1/2 = 0$. The (complex) roots are $\lambda_{1,2} = 1/2 \pm i/2$. Both have modulus $\rho = \sqrt{1/4 + 1/4} = 1/\sqrt{2} < 1$, hence the system is g.a.s. and the limit of any trajectory is the equilibrium point,

$$X^0 = \begin{pmatrix} 1 - 1 & 0 - (-1/2) \\ 0 - 1 & 1 - 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Example 5.7. Study the stability of the system

$$x_{t+1} = x_t + 3y_t,$$
$$y_{t+1} = x_t/2 + y_t/2.$$

Solution: The matrix of the system is $\begin{pmatrix} 1 & 3 \\ 1/2 & 1/2 \end{pmatrix}$, with characteristic equation $\lambda^2 - (3/2)\lambda - 1 = 0$. The roots are $\lambda_1 = 2$ and $\lambda_2 = -1/2$. The system is not g.a.s. However, there are initial conditions X_0 such that the trajectory converges to the fixed point $X^0 = (0, 0)$. This can be seen once we find the solution

$$X_t = PD^tP^{-1}X_0.$$

The eigenspaces are $S(2) =< (3, 1) >$ and $S(-1/2) =< (2, -1) >$, thus

$$P = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 2^t\frac{3}{5}(x_0 + 2y_0) + 2^{1-t}\frac{1}{5}(x_0 - 3y_0) \\ 2^t\frac{1}{5}(x_0 + 2y_0) - 2^{-t}\frac{1}{5}(x_0 - 3y_0) \end{pmatrix}.$$

If the initial conditions are linked by the relation $x_0 + 2y_0 = 0$, then the solution converges to $(0, 0)$. For this reason, the line $x + 2y = 0$ is called the *stable manifold.* Notice that the stable manifold is in fact the eigenspace associated to the eigenvalue $\lambda_2 = -1/2$, since

$$S(-1/2) =< (2, -1) >= \{x + 2y = 0\}.$$

For any other initial condition $(x_0, y_0) \notin S(-1/2)$, the solution does not converge.

Example 5.8 (Dynamic Cournot adjustment)**.** The purpose of this example is to investigate under what conditions a given adjustment process converges to the Nash equilibrium of the Cournot game.

Consider a Cournot duopoly in which two firms produce the same product and face constant marginal costs $c_1 > 0$ and $c_2 > 0$. The market price P_t is a function of the total quantity of output produced $Q = q_1 + q_2$ in the following way

$$P = \alpha - \beta Q, \qquad \alpha > c_i, \quad i = 1, 2, \quad \beta > 0.$$

In the Cournot duopoly model each firm chooses q_i to maximize profits, taking as given the production level of the other firm, q_j. At time t, firm i's profit is

$$\pi_i = q_iP - c_iq_i.$$

As it is well–known, taking $\partial\pi^i/\partial q_i = 0$ we obtain the best response of firm i, which depends on the output of firm j as follows[1]

$$\mathrm{br}_1 = a_1 - q_2/2, \qquad \mathrm{br}_2 = a_2 - q_1/2,$$

[1] Actually, the best response map is $\mathrm{br}_i = \max\{a_i - q_j/2, 0\}$, since negative quantities are not allowed.

where $a_i = \dfrac{\alpha - c_i}{2\beta}$, $i = 1, 2$. We suppose that $a_1 > a_2/2$ and that $a_2 > a_1/2$ in order to have positive quantities in equilibrium, as will be seen below.

The Nash equilibrium of the game, (q_1^N, q_2^N), is a pair of outputs of the firms such that none firm has incentives to deviate from it unilaterally, that is, it is the best response against itself. This means that the Nash equilibrium of the static game solves

$$\begin{aligned} q_1^N &= \mathrm{br}_1(q_2^N), \\ q_2^N &= \mathrm{br}_2(q_1^N). \end{aligned}$$

In this case

$$\begin{aligned} q_1^N &= a_1 - q_2^N/2, \\ q_2^N &= a_2 - q_2^N/2. \end{aligned}$$

Solving, we have

$$\begin{aligned} q_1^N &= \frac{4}{3}\left(a_2 - \frac{a_1}{2}\right), \\ q_2^N &= \frac{4}{3}\left(a_1 - \frac{a_2}{2}\right), \end{aligned}$$

which are both positive by assumption. As a specific example, suppose for a moment that the game is symmetric, with $c_1 = c_2 = c$. Then, $a_1 = a_2 = \dfrac{\alpha - c}{2\beta}$ and the Nash equilibrium is the output

$$\begin{aligned} q_1^N &= \frac{\alpha - c}{3\beta}, \\ q_2^N &= \frac{\alpha - c}{3\beta}. \end{aligned}$$

Now we turn to the general asymmetric game and introduce a dynamic component in the game as follows. Suppose that each firm does not choose its Nash output instantaneously, but they adjust gradually its output q_i towards its best response br_i at each time t as indicated below

$$\left\{ \begin{aligned} q_{1,t+1} &= q_{1,t} + d(\mathrm{br}_{1,t} - q_{1,t}) = q_{1,t} + d(a_1 - \tfrac{1}{2} q_{2,t} - q_{1,t}), \\ q_{2,t+1} &= q_{1,t} + d(\mathrm{br}_{2,t} - q_{2,t}) = q_{2,t} + d(a_2 - \tfrac{1}{2} q_{1,t} - q_{2,t}), \end{aligned} \right. \tag{5.3}$$

where d is a positive constant. The objective is to study whether this *tattonement* process converges to the Nash equilibrium.

To simplify notation, let us rename $x = q_1$ and $y = q_2$. Then, rearranging terms in the system (5.3) above, it can be rewritten as

$$\left\{ \begin{aligned} x_{t+1} &= (1-d)x_t - \tfrac{d}{2} y_t + d a_1, \\ y_{t+1} &= (1-d)y_t - \tfrac{d}{2} x_t + d a_2. \end{aligned} \right.$$

It is easy to find the equilibrium points by solving the system

$$\begin{cases} x = (1-d)x - \frac{d}{2}y + da_1, \\ y = (1-d)y - \frac{d}{2}x + da_2. \end{cases}$$

The only solution is precisely the Nash equilibrium,

$$(x^N, y^N) = \left(\frac{4}{3}\left(a_2 - \frac{a_1}{2}\right), \frac{4}{3}\left(a_1 - \frac{a_2}{2}\right) \right).$$

Under what conditions this progressive adjustment of the produced output does converge to the Nash equilibrium? According to the theory, it depends on the module of the eigenvalues being smaller than 1. Let us find the eigenvalues of the system. The matrix of the system is

$$\begin{pmatrix} 1-d & -\frac{d}{2} \\ -\frac{d}{2} & 1-d \end{pmatrix}$$

The eigenvalues of the matrix are

$$\lambda_1 = 1 - \frac{d}{2}, \qquad \lambda_2 = 1 - \frac{3d}{2},$$

which only depend on d. We have

$$|\lambda_1| < 1 \text{ iff } 0 < d < 4,$$
$$|\lambda_2| < 1 \text{ iff } 0 < d < 4/3,$$

therefore $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff $0 < d < 4/3$. Thus, $0 < d < 4/3$ is a necessary and sufficient condition for convergence to the Nash equilibrium of the one shot game from any initial condition (g.a.s. system).

6. The nonlinear first order equation

We investigate here the stability of the solutions of an autonomous first order difference equation

$$x_{t+1} = f(x_t), \qquad t = 0, 1, \dots,$$

where $f : I \to I$ is nonlinear and I is an interval of the real line. Recall that a function f is said to be *of class* C^1 in an open interval, if f' exists and it is continuous in that interval. For example, the functions x^2, $\cos x$ or e^x are C^1 is the whole real line, but $|x|$ is not differentiable at 0, so is not C^1 in any open interval that contains 0.

Theorem 6.1. *Let $x^0 \in I$ a fixed point of f, and suppose that f is C^1 in an open interval around x^0, $I_\delta = (x^0 - \delta, x^0 + \delta)$.*

(1) If $|f'(x^0)| < 1$, then x^0 is locally asymptotically stable;
(2) If $|f'(x^0)| > 1$, then x^0 is unstable.

Remark 6.2. If $|f'(x)| < 1$ for every point $x \in I$, then the fixed point x^0 is globally asymptotically stable.

Example 6.3 (Population growth models)**.** In the Malthus model of population growth it is postulated that a given population x grows at constant rate r,

$$\frac{x_{t+1} - x_t}{x_t} = r, \qquad \text{or} \qquad x_{t+1} = (1+r)x_t.$$

This is a linear equation and the population grows unboundedly if the per capita growth rate r is positive[2]. This is not realistic for large t. When the population is small, there are ample environmental resources to support a high birth rate, but for later times, as the population grows, there is a higher death rate as individuals compete for space and food. Thus, the growth rate should be decreasing as the population increases. The simplest case is to take a linearly decreasing per capita rate, that is

$$r\left(1 - \frac{x_t}{K}\right),$$

where K is the carrying capacity. This modification is known as the *Verhulst' law*. Then the population evolves as

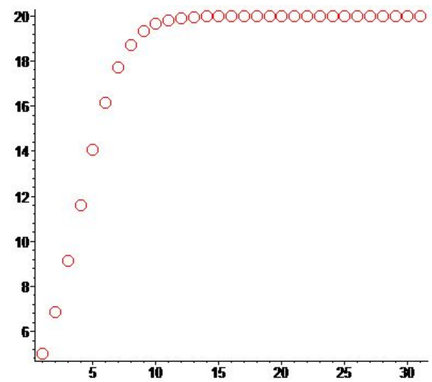
$$x_{t+1} = x_t\left(1 + r - \frac{r}{K}x_t\right),$$

which is not linear. The function f is quadratic, $f(x) = x(1 + r - rx/K)$. In Fig. 6.3 is depicted a solution with $x_0 = 5$, $r = 0.5$ and $K = 20$.

We observe that the solution converges to 20. In fact, there are two fixed points of the equation, 0 (extinction of the population) and $x^0 = K$ (maximum carrying capacity). Considering the derivative of f at these two fixed points, we have

$$f'(0) = 1 + r - 2\frac{r}{K}x\Big|_{x=0} = 1 + r > 1,$$
$$f'(K) = 1 + r - 2\frac{r}{K}x\Big|_{x=K} = 1 - r.$$

[2]The solution is $x_t = (1+r)^t x_0$, why?



Thus, according to Theorem 6.1, 0 is unstable, but K is l.a.e. iff $|1-r|<1$, or $0<r<2$.

6.1. **Phase diagrams.** The stability of a fixed point of the equation

$$x_{t+1} = f(x_t), \qquad t = 0, 1, \ldots,$$

can also be studied by a graphical method based in *the phase diagram.* This consists in drawing the graph of the function $y = f(x)$ in the plane xy. Note that a fixed point x^0 corresponds to a point (x^0, x^0) where the graph of $y = f(x)$ intersects the straight line $y = x$.

The following figures show possible configurations around a fixed point. The phase diagram is at the left (plane xy), and a solution sequence is shown at the right (plane tx). Notice that we have drawn the solution trajectory as a continuous curve because it facilitates visualization, but in fact it is a sequence of discrete points. In Fig. 1, $f'(x^0)$ is positive, and the sequence $x_0, x_1, \ldots$ converges monotonically to x^0, whereas in Fig. 2, $f'(x^0)$ is negative and we observe a cobweb–like behavior, with the sequence $x_0, x_1, \ldots$ converging to x^0 but alternating between values above and below the equilibrium. In Fig. 3, the graph of f near x^0 is too steep for convergence. After many iterations in the diagram, we observe an erratic behavior of the sequence $x_0, x_1, \ldots$. There is no cyclical patterns and two sequences generated from close initial conditions depart along time at an exponential rate (see Theorem 6.1 above). It is often said that the equation exhibits chaos. Finally, Fig. 4 is the phase diagram of an equation admitting a cycle of period 3.

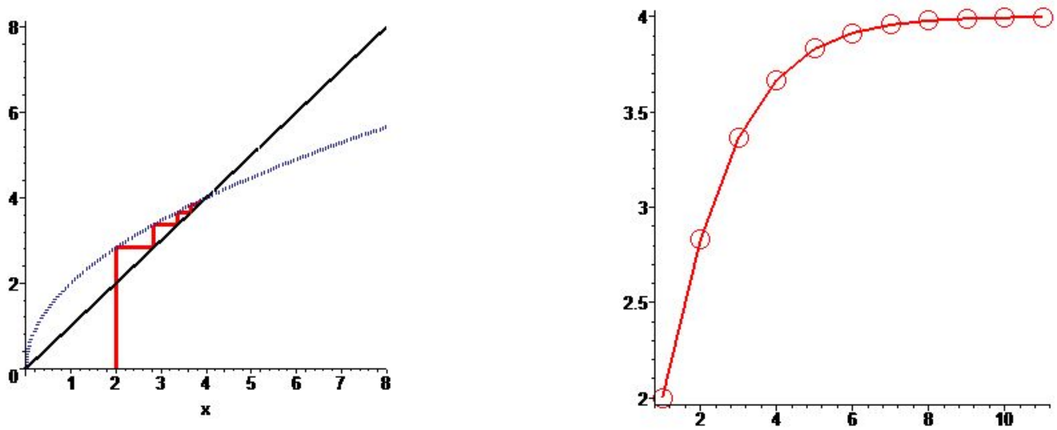


FIGURE 1. x^0 stable, $f'(x^0) \in (0, 1)$

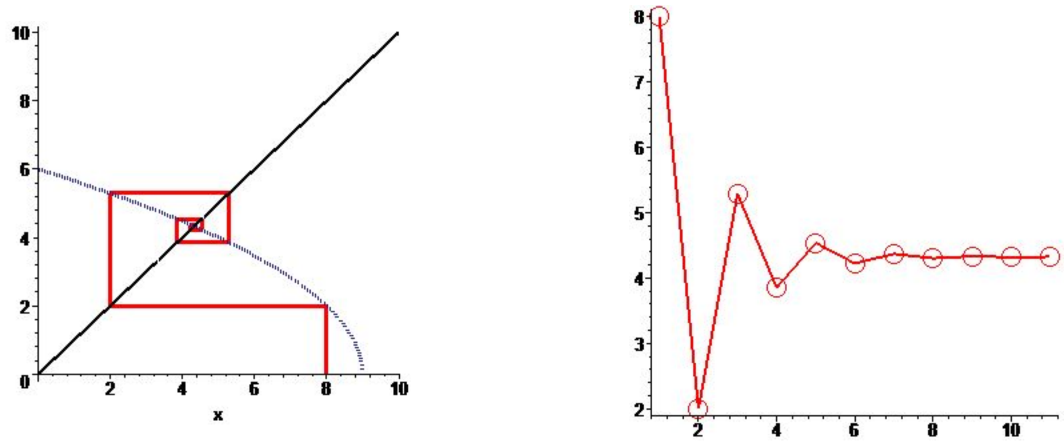


FIGURE 2. x^0 stable, $f'(x^0) \in (-1, 0)$

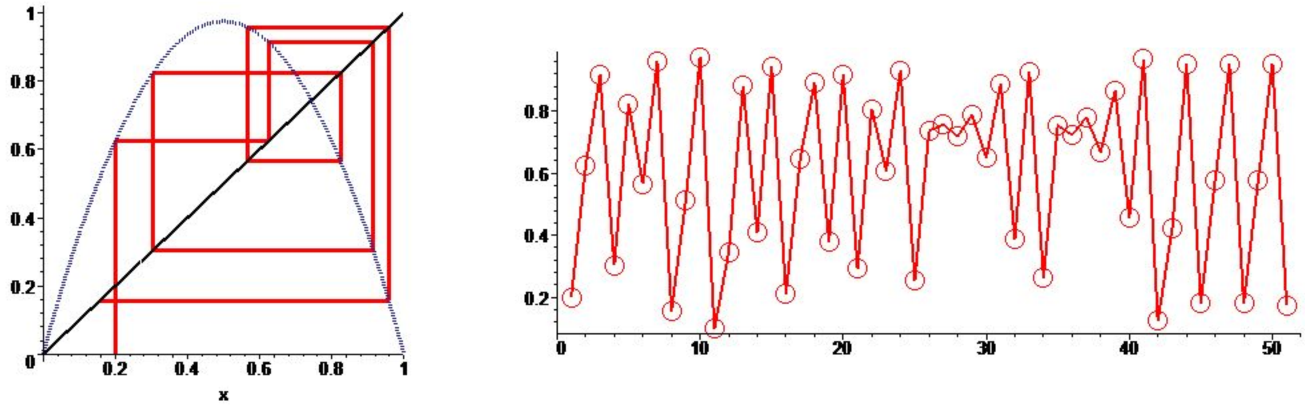


FIGURE 3. x^0 unstable, $|f'(x^0)| > 1$

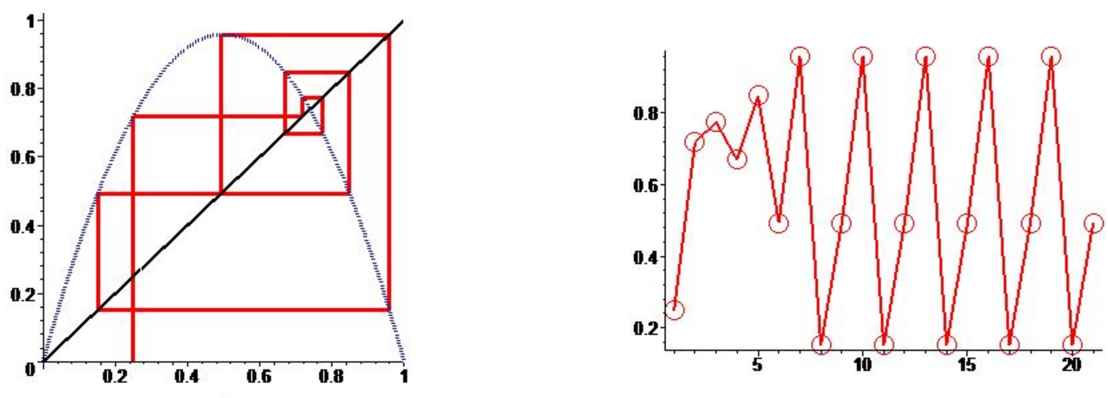


FIGURE 4. A cycle of period 3