Auctions

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What is an Auction?

Auction: (Latin: auctus, past participle of augēre: to increase.)
1. Merriam-Webster: A sale of property to the highest bidder.
2. Encyclopædia Britannica: The buying and selling of real and personal property through open public bidding.
3. McAfee-McMillan (1987): A market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants.

In Spanish:
Subasta. RAE: Del lat., sub hasta, bajo la lanza, porque la venta del botín cogido en la guerra se anunciaba con una lanza.
1. Venta pública de bienes o alhajas que se hace al mejor postor, y regularmente por mandato y con intervención de un juez u otra autoridad.
2. Adjudicación de una contrata, generalmente de servicio público, como la ejecución de una obra, el suministro de provisiones, etc., a quien presenta la propuesta más ventajosa.

History

▷Babylon (500 BC): sale of wives.

▷Roman Empire: sale of slaves, war booty, debtors' property, ... the whole Empire (Didius Julianus, emperor between April 28 and June 1, 193 AD).

▷ Recent Time: art and antiques, flowers, fish, agricultural products ... – Stockholm's Auktionsverk (1674), Sothebys (1744), Christies (1766).

▷Today: treasury bills, procurement, mineral rights, spectrum licences, real estate, firms (hostile takeovers), pollution permits, ... internet auctions, ... as much as 30% of GDP is contracted through auctions.

Common Auctions

- Open ascending price auctions (English Auctions).
- Open descending price auctions (Dutch auctions).
- First-price sealed-bid auction.
- Second-price sealed-bid auction (Vickrey Auctions).

	Open	Sealed-Bid
First-Price	DA	FPA
Second-Price	EA	SPA

Other Auctions

- Hybrid Anglo-Dutch auction (EA, then a FPA with the two finalists).
- Clock auctions (with or without a buy-now price).
- Candle auctions (random stopping time).
- Third-price auctions, ...
- All-pay auctions.

(All these are examples of a class of auctions known as *standard* – to be defined formally.)

Why using an auction?

Practical Reasons: an auction is a way to implement a market.

An auction provides the means for:

- price discovery (the seller and/or buyers may not know what the item or service is worth)
- winner determination
- payment mechanism.

Why using an auction?

Normative Reasons: auctions have good properties.

Well designed auctions:

- produce efficient outcomes
- maximize revenue
- are perceived as fair and transparent
- may prevent corruption
- etc.

Why using an auction?

Under perfect information,

- direct bargaining, or
- posted prices

may provide a simpler way to arrange trade.

Auctions are useful precisely because the seller and the buyers are unsure about the values of the objects up for sale.

The IPV Model

There is a single object for sale.

- The object has a pure consumption (private) value (e.g., a rare wine).[1]
- There are n risk neutral buyers (bidders).
- Bidders' values $(X_1, ...X_n)$ are iid according to some increasing differentiable cdf F whose support is an interval $[0, \omega]$.
- Each bidder knows her value, but is uncertain about the values of the other bidders (i.e., values are private information).

[1] When an object has an investment value, perhaps in addition to its consumption value, bidders have *interdependent* values. For example, the right to extract the oil of a track of land.

Auctions

In an auction:

- Each bidder i places a bid $b_i \in \mathbb{R}_+$.
- The profile of bids $(b_1, ..., b_n)$ determines the probability with which each bidder wins the object, $p_i(b_1, ..., b_n)$, and how much each bidder pays, $t_i(b_1, ..., b_n)$.

An auction (p, t), where $p : \mathbb{R}^n_+ \to \Delta^n$ and $t : \mathbb{R}^n_+ \to \mathbb{R}^n$, defines a *game of incomplete information*.

An Auction Game

In the Bayesian game $\Gamma = (N, T, A, u, \mu)$ defined by an auction (p, t):

- The players are the bidders, i.e., $N = \{1, ..., n\}$.
- A player's type is her value, i.e., $T_i = [0, \omega]$.
- A player's action is a bid, i.e., $A_i = \mathbb{R}_+$
- The payoff of a player $u_i : [0, \omega]^n \times \mathbb{R}^n_+ \to \mathbb{R}$ is

$$u_i(x, b) = p_i(b)x_i - t_i(b).$$

- Players' beliefs are described by the c.d.f.

$$\mu(x_1, .., x_n) = \prod_{i=1}^{n} F(x_i).$$

Strategically Equivalent Auctions

In the context of the IPV model:

- The English action is *strategically equivalent* to the second-price sealed-bid auction.
- The Dutch action is *strategically equivalent* to the first-price sealed-bid auction.

SPA - Equilibrium Analysis

In a Second Price Auction:

- The highest bidder wins the object
- The winner pays the second highest bid, and the losers pay nothing.
- Ties are resolved by, e.g., assigning the object to the highest bidders with equal probability.

The functions (p, t) defining a second-price auction are given for $b \in \mathbb{R}^n_+$ by

$$p_i^{SPA}(b) = \begin{cases} \frac{1}{|M(b)|} & \text{if } b_i \in M(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$t_i^{SPA}(b) = p_i^{SPA}(b) \max_{j \in N \setminus \{i\}} b_j,$$

where $M(b) = \{j \in N \mid b_j = \max_{k \in N} b_k\}$.

SPA - Equilibrium Analysis

The Bayesian game defined by a SPA in the IPV setting has a continuum of equilibria. However, ...

Prop. 1. *In a SPA, true value bidding, i.e.,* $\beta(x) = x$*, is a weakly dominant strategy.*

Discuss a heuristic proof.

True value bidding is a *natural* equilibrium selection due to its strategic simplicity, symmetry, etc. We identify the outcome generated by a SPA as that resulting from this equilibrium.

SPA - Equilibrium Analysis

Some basic properties of the outcome of a SPA are direct corollaries of Prop. 1:

Corollary 1. *In a SPA,*

(1.1) The bidder with the highest value wins the object.

(1.2) The seller revenue is the second largest value.

Remark 1. *Under independent private values the results above extend to English actions.*

Order Statistics

Let $X_1, ..., X_k$ be iid on $[0, \omega]$ according to F. For $j \leq k$ let $Y_j^{(k)}$ be the jth largest value of any realization of $X_1, ..., X_k$.

The cdf of $Y_1^{(k)}$, denoted $G_1^{(k)}$, is

$$\begin{aligned} G_1^{(k)}(y) &= \Pr(Y_1^{(k)} \leq y) \\ &= \Pr(X_1 \leq y, ..., X_k \leq y) \\ \text{(Independence)} &= F(y)^k. \end{aligned}$$

Order Statistics

The cdf of $Y_2^{(k)}$, denoted $G_2^{(k)}$, is

$$\begin{aligned}
G_2^{(k)}(y) &= \Pr(Y_2^{(k)} \leq y) \\
\text{(see Figure 1 below)} &= \Pr\left(X_1 \leq y, X_2 \leq y, ..., X_k \leq y\right) \\
&\quad + \Pr\left(X_1 > y, X_2 \leq y, ..., X_k \leq y\right) \\
&\quad ... + \Pr\left(X_1 \leq y, X_2 \leq y, ..., X_k > y\right) \\
&= F(y)^k + k\left(1 - F(y)\right) F(y)^{k-1} \\
&= kF(y)^{k-1} - (k-1)F(y)^k.
\end{aligned}$$

SPA - Equilibrium Analysis

$X_1 \leq y$	$X_1 > y$	$X_2 \leq y$	$\ldots$	$X_k \leq y$
$\vdots$			$\vdots$	
$X_k \leq y$	$X_1 \leq y$	$X_2 \leq y$	$\ldots$	$X_k > y$

Figure 1: Partition of the Event $Y_2^{(k)} \leq y$.

SPA - Equilibrium Analysis

Payoffs and Revenue in a SPA

By Corollary 1 calculating the seller's expected revenue and the bidders' expected payoffs in a SPA is a simple task.

The seller revenue R^{SPA} is equal to $Y_2^{(n)}$. Therefore the expected seller revenue is

$$\bar{R}^{SPA} = E(Y_2^{(n)}).$$

The expected payoff of a bidder whose value is x is

$$U^{SPA}(x) = \Pr(\text{winning} \mid x)[x - E(\text{Second Highest Bid} \mid x)].$$

Since a bidder wins the auction when her value is the largest, then

$$\Pr(\text{winning} \mid x) = \Pr(Y_1^{(n-1)} < x) = G_1^{(n-1)}(x).$$

SPA - Equilibrium Analysis

Payoffs and Revenue in a SPA

Since all bidders bid their value, the expected payment of a bidder who wins the auction when her value is x is

$$E(\text{Second Highest Bid} \mid x) = E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

Hence

$$U^{SPA}(x) = G_1^{(n-1)}(x)[x - E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x)].$$

Therefore the ex-ante expected payoff of a bidder is

$$\bar{U}^{SPA} = E(U^{SPA}(X)).$$

Surplus in a SPA

For each realization of values, the gross surplus generated in the action,[2] denoted S^{SPA}, is the largest value; i.e., S^{SPA} is the order statistic $Y_1^{(n)}$. Hence the expected gross surplus is

$$\bar{S}^{SPA} = E(Y_1^{(n)}).$$

(**Exercise**. Show that

$$\bar{S}^{SPA} = \bar{R}^{SPA} + n\bar{U}^{SPA}.$$

Hint. The expected payment of a bidder whose value is x is

$$m^{SPA}(x) = G_1^{(n-1)}(x)E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

Hence

$$\bar{R}^{SPA} = nE\left(m^{SPA}(X)\right).$$

SPA - Equilibrium Analysis

Example

Let $n = 2$, $\omega = 1$, $F(x) = x$ (uniform). The seller revenue is $Y_2^{(2)}$ whose cdf on $[0, 1]$ is

$$G_2^{(2)}(y) = 2F(y) - F(y)^2 = 2y - y^2.$$

Hence the expected seller revenue is

$$\bar{R}^{SPA} = \int_0^1 y dG_2^{(2)}(y) = 2\int_0^1 y\,(1-y)\,dy = \frac{1}{3}.$$

The expected payment of a bidder whose value is x is

$$m^{SPA}(x) = G_1^{(1)}(x)E(Y_1^{(1)} \mid Y_1^{(1)} < x) = \frac{x^2}{2}.$$

where

$$G_1^{(1)}(y) = F(y) = y,$$

and

$$E(Y_1^{(1)} \mid Y_1^{(1)} < x) = E(Y \mid Y < x) = \frac{x}{2}.$$

Example

Hence the expected payoff of a bidder whose value is x is

$$U^{SPA}(x) = G_1^{(1)}(x)x - m^{SPA}(x) = \frac{x^2}{2},$$

and her ex-ante expected payoff is

$$\bar{U}^{SPA} = E\left(U^{SPA}(X)\right) = \int_0^1 \frac{x^2}{2}dx = \frac{1}{6}.$$

Finally, the gross surplus is $Y_1^{(2)}$, where $G_1^{(2)}(y) = y^2$, and therefore the expected gross surplus is

$$\bar{S}^{SPA} = E(Y_1^{(2)}) = \int_0^{\omega} ydG_1^{(2)}(y) = 2\int_0^1 y^2dy = \frac{2}{3}.$$

FPA - Equilibrium Analysis

In a FPA:

- The highest bidder wins the object.
- The winner pays her bid, and the losers pay nothing.

(Ties are resolved by, e.g., assigning the object to the highest bidders with equal probability. But how ties are resolved does not affect equilibrium.)

The functions (p, t) defining a first-price auction are given for $b \in \mathbb{R}^n_+$ by

$$p_i^{FPA}(b) = \begin{cases} \frac{1}{|M(b)|} & \text{if } b_i \in M(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$t_i^{FPA}(b) = p_i^{FPA}(b) b_i.$$

where as before $M(b) = \{j \in N \mid b_j = \max_{k \in N} b_k\}$.

FPA - Equilibrium Analysis

In the incomplete information game defined by a FPA, true value bidding is neither a weakly dominant strategy nor an equilibrium.

However, there is a unique *symmetric increasing differentiable* equilibrium strategy β.

Prop. 2. (Vickrey, 1961) *The FPA has a unique symmetric equilibrium* $(\beta, ..., \beta)$ *such that* β *is differentiable and increasing. This equilibrium is given by*

$$\beta(x) = E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

Remark 2. *This result extends to DA, even if values are interdependent.*

Proof of Prop 2

Let $(\beta, ..., \beta)$ be a symmetric equilibrium such that $\beta' > 0$.

1. $\beta(0) = 0$.

2. Bidding $b > \beta(\omega)$ is suboptimal.

3. For each $x \in [0, \omega]$ bidding $\beta(x)$ must be optimal. Since the expected payoff of a bidder who bids $b \leq \beta(\omega)$ when her value is $x \in [0, \omega]$ is

$$G_1^{(n-1)}(\beta^{-1}(b))\,(x - b)\,,$$

where $\beta^{-1}(b)$ is the value in $[0, \omega]$ that bids b. Then $b = \beta(x)$ solves

$$\max_{b \in [0, \beta(\omega)]} G_1^{(n-1)}(\beta^{-1}(b))\,(x - b)\,.$$

FPA - Equilibrium Analysis

Proof of Prop 2 (cont.)

3. Set $z = \beta^{-1}(b)$; then $z = x$ solves

$$\max_{z \in [0,\omega]} G_1^{(n-1)}(z)\left(x - \beta(z)\right).$$

Since both F and β are differentiable, x solves

$$g_1^{(n-1)}(x)\left(x - \beta(x)\right) - G_1^{(n-1)}(x)\beta'(x) = 0,$$

i.e.,

$$G_1^{(n-1)}(x)\beta'(x) + g_1^{(n-1)}(x)\beta(x) = x g_1^{(n-1)}(x),$$

or,

$$\frac{d}{dx}\left(G_1^{(n-1)}(x)\beta(x)\right) = x g_1^{(n-1)}(x).$$

Proof of Prop 2 (cont.)

Hence

$$G_1^{(n-1)}(x)\beta(x) = \int_0^x y g_1^{(n-1)}(y)dy,$$

i.e.,

$$\beta(x) = \int_0^x y \frac{g_1^{(n-1)}(y)}{G_1^{(n-1)}(x)} dy.$$

Proof of Prop 2 (cont.)

The cdf $J_x(y)$ of the random variable $Y_1^{(n-1)} \mid Y_1^{(n-1)} < x$ is

$$\begin{aligned} J_x(y) &= \Pr(Y_1^{(n-1)} < y \mid Y_1^{(n-1)} < x) \\ &= \frac{\Pr(Y_1^{(n-1)} < y, Y_1^{(n-1)} < x)}{\Pr(Y_1^{(n-1)} < x)}. \end{aligned}$$

Hence for $y > x$, we have $J_x(y) = 0$, and for $y \leq x$ we have

$$J_x(y) = \frac{G_1^{(n-1)}(y)}{G_1^{(n-1)}(x)}.$$

Thus

$$\beta(x) = \int_0^x y \frac{g_1^{(n-1)}(y)}{G_1^{(n-1)}(x)} dy = \int_0^\omega y dJ_x(y) = E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

Proof of Prop 2 (cont.)

4. So far we have shown that if there is a symmetric increasing differentiable equilibrium, then the bidding strategy is

$$\beta(x) = E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

In order to complete the proof of Prop. 2 we need to show that $(\beta, ..., \beta)$ is indeed an equilibrium.

We show that $\beta(x)$ is optimal for each $x \in [0, \omega]$, i.e., that x solves

$$\max_{z \in [0,\omega]} G_1^{(n-1)}(z)\,(x - \beta(z)) = G_1^{(n-1)}(z)x - \int_0^z y g_1^{(n-1)}(y)dy.$$

Proof of Prop 2 (cont.)

Integration by parts ($u(y) = y$, $v(y) = G_1^{(n-1)}(y)$) yields

$$\int_0^z y g_1^{(n-1)}(y) dy = G_1^{(n-1)}(z) z - \int_0^z G_1^{(n-1)}(y) dy$$

Hence

$$G_1^{(n-1)}(z) \left(x - \beta(z) \right) = G_1^{(n-1)}(z) \left(x - z \right) + \int_0^z G_1^{(n-1)}(y) dy.$$

and

$$G_1^{(n-1)}(x) \left(x - \beta(x) \right) = \int_0^x G_1^{(n-1)}(y) dy.$$

FPA - Equilibrium Analysis

Proof of Prop 2 (cont.)
Thus for all $z \in [0, \omega]$ we have

$$\begin{aligned} G_1^{(n-1)}(z)\,(x-\beta(z)) - G_1^{(n-1)}(x)\,(x-\beta(x)) &= G_1^{(n-1)}(z)\,(x-z) \\ &\quad - \int_z^x G_1^{(n-1)}(y)dy. \end{aligned}$$

Since G is non-decreasing, for $z < x$

$$\int_z^x G_1^{(n-1)}(y)dy \geq G_1^{(n-1)}(z) \int_z^x dy = G_1^{(n-1)}(z)\,(x-z)\,,$$

and for $z > x$.

$$-\int_z^x G_1^{(n-1)}(y)dy = \int_x^z G_1^{(n-1)}(y)dy \leq G_1^{(n-1)}(z)\,(z-x)\,.$$

Proof of Prop 2 (cont.)

Hence for all $z \in [0, \omega]$

$$G_1^{(n-1)}(z)\,(x - \beta(z)) - G_1^{(n-1)}(x)\,(x - \beta(x)) \leq 0,$$

and therefore $z = x$ (equivalently, bidding $\beta(x)$) is optimal.

Value Shading in a FPA

Integrating $\beta(x)$ by parts (choosing $u(y) = y$ and $v(y) = G_1^{(n-1)}(y)/G_1^{(n-1)}(x)$) yields

$$\begin{aligned}
\beta(x) &= \int_0^x y \frac{g_1^{(n-1)}(y)}{G_1^{(n-1)}(x)} dy \\
&= x - \int_0^x \frac{G_1^{(n-1)}(y)}{G_1^{(n-1)}(x)} dy \\
&= x - \int_0^x \left(\frac{F(y)}{F(x)} \right)^{n-1} dy < x.
\end{aligned}$$

That is, in a FPA bidders bid below value. The extend to which bidders *shade* their bids depends on the distribution of values and the number of bidders.

Payoffs and Revenue in a FPA

In a FPA, the expected payoff of a bidder whose value is x is

$$\begin{aligned} U^{FPA}(x) &= \Pr(\text{winning} \mid x)\,(x - \beta(x)) \\ &= G_1^{(n-1)}(x)\left(x - E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x)\right) \\ &= U^{SPA}(x). \end{aligned}$$

Hence bidders expected payoff are the same in FPA and SPA.
Since a in FPA bidder pays her bid, the expected payment of a bidder whose value is x is

$$m^{FPA}(x) = G_1^{(n-1)}(x)E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x) = m^{SPA}(x).$$

Payoffs and Revenue in a FPA

Since $m^{FPA}(x) = m^{SPA}(x)$, the seller expected revenue in a FPA is

$$\bar{R}^{FPA} = nE(m^{FPA}(x)) = nE(m^{SPA}(x)) = \bar{R}^{SPA}.$$

However, the seller revenue in a FPA R^{FPA} is the r.v. $\beta(Y_1^{(n)})$, whereas his revenue in a SPA R^{SPA} is the r.v. $Y_2^{(n)}$. Thus, for a given realization of bidders' values the seller revenue in a FPA and SPA generally differs.

Revenue in FP and SP Auctions

Revenue Comparison in a FPA and a SPA

In fact, the distribution of the seller revenue in a FPA is more concentrated around the mean than in a SPA:

Prop. 3. *The seller revenue in a SPA is a mean preserving spread of the seller revenue in a FPA.*

Example

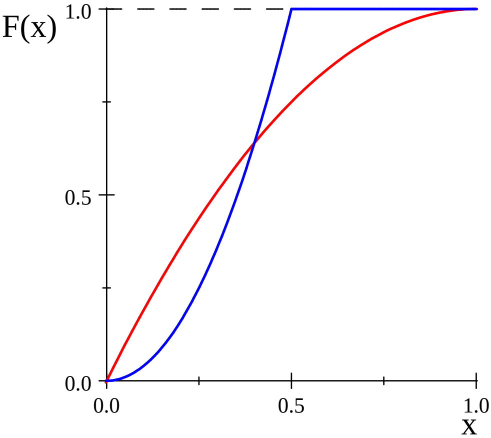
Let $n = 2$, $\omega = 1$, $F(x) = x$ (uniform). Then, in a *FPA* bidders bid according to

$$\beta^{FPA}(x) = E(Y_1^{(1)} \mid Y_1^{(1)} < x) = \frac{x}{2}.$$

Note that the cdf of $Y_2^{(2)}$ is $G_2^{(2)}(y) = 2y - y^2$, and its support is $[0, 1]$, whereas the cdf of $\beta^{FPA}(Y_1^{(2)})$ is $H(y) = 4y^2$ and its support is $[0, \frac{1}{2}]$.

(See the figure below.)

Revenue Comparison in a FPA and a SPA



The cdfs of R^{FPA} (in blue) and R^{SPA} (in red).

FP and SP Auctions: pros and cons

SPA:

- $\oplus$ Strategic simplicity.
- $\oplus$ Efficiency (even if values are asymmetric).
- $\ominus$ Susceptible to corruption, political embarrassment.[3]
- $\ominus$ Susceptible to collusion.

FPA:

- $\ominus$ Strategic sophistication.
- $\oplus\ominus$ Efficiency (but not if values are asymmetric).
- $\oplus\ominus$ Less susceptible to corruption, political embarrassment.
- $\oplus\ominus$ Less susceptible to collusion.

[3] Values are reveled: New Zeland spectrum auctions.

Revenue Equivalence

The expected seller revenue is the same in a FPA and SPA.

This revenue equivalence extends to any standard auction.

(An auction is *standard* if it allocates the object to the highest bidder and treats bids equally. In particular, it involves no probabilistic allocation of the item – except when there are ties.)

Example

Let $n = 2$, $\omega = 1$, $F(x) = x$ (uniform). Let β^{APA} be an increasing symmetric equilibrium of the all pay auction such that $m^{APA}(0) = 0$. Since a bidders pays her bid whether she wins or not, then

$$\beta^{APA}(x) = m^{APA}(x) = G_1^{(n-1)}(x)E(Y_1^{(n-1)} \mid Y_1^{(n-1)} < x).$$

In our example

$$\beta^{APA}(x) = G_1^{(1)}(x)E(Y_1^{(1)} \mid Y_1^{(1)} < x) = \frac{x^2}{2}.$$

(Compare $\beta^{FPA}(x) = x/2$ and $\beta^{SPA}(x) = x$.)

Reserves Prices and Entry Fees

- By the Revenue Equivalence Theorem a (risk neutral) seller whose objective is to maximize his expected revenue would be indifferent between any standard auction. However, the seller may have a value for the object. In this case, the payoff to the seller differs from the expected revenue.
- The payoff of a risk neutral seller with value $x_0 \in (0, \omega)$ who sells the object using a standard auction is equal to the revenue minus x_0. Also the net social surplus is equal to the gross surplus minus the seller value.
- A disadvantage of selling the object using a standard auction is that the object is allocated to the bidder with the largest value, even when this value is below the seller's value.

Reserves Prices and Entry Fees

- A way to avoid this problem (and to improve the efficiency of the outcome) is to introduce (and to announce publicly) a reserve price below which the object remains with the seller.

- In an auction with a reserve price $r \in (0, \omega)$, bids below r are ignored.

- How does a reserve price affect the outcome of an auction?

The Revenue Equivalence Theorem
extends to auctions with reserve prices and/or entry fees.

Seller Expected Revenue and Payoff

The seller expected revenue in an standard auction with a reserve price $r \in [0, \omega]$ is

$$\bar{R}(r) = nE(m(X, r)),$$

and her expected payoff is

$$\begin{aligned} \bar{V}(r) &= \bar{R}(r) + x_0 G_1^{(n)}(r) \\ &= nE(m(X, r)) + x_0 G_1^{(n)}(r). \end{aligned}$$

Hence

$$\bar{V}'(r) = n\frac{dE(m(X, r))}{dr} + x_0 g_1^{(n)}(r).$$

Seller Expected Revenue and Payoff

$$
\begin{aligned}
E(m(X,r)) &= \int_r^{\omega}\left(rG_1^{(n-1)}(r)+\int_r^{x} yg_1^{(n-1)}(y)dy\right)dF(x)\\
&= r\left(1-F(r)\right)G_1^{(n-1)}(r)+\int_r^{\omega}\int_r^{x} yg_1^{(n-1)}(y)f(x)dxdy\\
&= r\left(1-F(r)\right)G_1^{(n-1)}(r)\\
&\quad+\int_r^{\omega}\left(\int_y^{\omega} f(x)dx\right)yg_1^{(n-1)}(y)dy\\
&= r\left(1-F(r)\right)G_1^{(n-1)}(r)\\
&\quad+\int_r^{\omega} y\left(1-F(y)\right)g_1^{(n-1)}(y)dy.
\end{aligned}
$$

Seller Revenue and Payoff

Taking derivatives (using Leibniz integral rule):

$$
\begin{aligned}
\frac{dE(m(X,r))}{dr} &= \frac{d}{dr}\left(rG_1^{(n-1)}(r)\left(1-F(r)\right)\right) \\
&\quad + \frac{d}{dr}\int_r^{\omega} y\left(1-F(y)\right)g_1^{(n-1)}(y)dy \\
&= \left(1-F(r)\right)G_1^{(n-1)}(r) \\
&\quad + r\left(\left(1-F(r)\right)g_1^{(n-1)}(r) - G_1^{(n-1)}(r)f(r)\right) \\
&\quad - r\left(1-F(r)\right)g_1^{(n-1)}(r) \\
&= G_1^{(n-1)}(r)\left(1-F(r)-rf(r)\right).
\end{aligned}
$$

Seller Revenue and Payoff

Then

$$\begin{aligned}\bar{V}'(r) &= nG_1^{(n-1)}(r)\,(1 - F(r) - rf(r)) + x_0 g_1^{(n)}(r) \\ &= nG_1^{(n-1)}(r)\,(1 - F(r) - rf(r) + x_0 f(r)) \\ &= nG_1^{(n-1)}(r)\,(1 - F(r) - (r - x_0)f(r))\,.\end{aligned}$$

Assuming $F(r) < 1$, we can rewrite this expression as

$$\bar{V}'(r) = nG_1^{(n-1)}(r)\,(1 - F(r))\,(1 - (r - x_0)\,\lambda(r))\,.$$

where

$$\lambda(x) = \frac{f(x)}{1 - F(x)}$$

is the hazard rate of F.

Seller Revenue and Payoff

Thus, the reserve price r^* that maximizes the seller's expected payoff solves

$$1 - (r - x_0)\,\lambda(r) = 0.$$

i.e.,

$$r^* = x_0 + \frac{1}{\lambda(r^*)}$$

Second Order Sufficent Condition: if λ is increasing, then

$$\bar{V}''(r^*) = -nG_1^{(n-1)}(r^*)\,(1 - F(r^*))\,\big(\lambda(r^*) + (r^* - x_0)\,\lambda'(r^*)\big) < 0.$$

(Although $\bar{V}'(0) = 0$, since

$$\bar{V}''(0) = ng_1^{(n-1)}(0)\,(1 + x_0 f(0)) > 0,$$

$r = 0$ is a minimum.)

Seller Revenue and Payoff

Interestingly, the reservation price that maximizes the seller's expected payoff is, $r^* = x_0 + 1/\lambda(r^*)$, is independent of the number of bidders n!

Thus, in a first- or second-price sealed bid auction the revenue maximizing reserve price does not depend on the number of bidders. This is because a reserve price is relevant only when the values of all but one bidder exceed the reserve.

Example

Let $n = 2$, $\omega = 1$, $F(x) = x$ (uniform). Calculate the revenue maximizing reserve price assuming that the seller's value is $x_0 = 0$.

The hazard rate is

$$\lambda(x) = \frac{1}{1-x}.$$

Hence the optimal reserve solves

$$r = 1 - r.$$

i.e.,

$$r^* = \frac{1}{2}.$$

Example

The seller expected payoff (which is equal to the expected seller revenue) with this reserve price is

$$
\begin{aligned}
\bar{V}(r^*) &= \bar{R}(r^*) = nE\left(m(X, r^*)\right) \\
&= nr^*\left(1 - F(r^*)\right) G_1^{(1)}(r^*) \\
&\quad + n\int_{r^*}^{\omega} y\left(1 - F(y)\right) g_1^{(1)}(y)dy \\
&= 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + 2\int_{\frac{1}{2}}^{1} y\left(1 - y\right) dy = \frac{5}{12}.
\end{aligned}
$$

Exercise. Assuming that the seller's value is $x_0 = 0$, calculate the optimal reserve price and the expected sellers revenue when (i) $n > 2$ and $F(x) = x$, (ii) $n = 2$ and $F(x) = x^2$, and (iii) $n = 2$ and $F(x) = 2x - x^2$.

Reserves Prices and Entry Fees

Entry Fees
In a first or second price auction, a reserve price r excludes from the auction all bidders with a value $x < r$.

Removing the reserve price r and imposing instead an *entry fee*

$$e = G_1^{(n)}(r)r$$

that each bidder must pay in order to participate in the auction effectively has the same effect; that is, *in a symmetric equilibrium, the bidders whose realized value is above r participate in the auction and bid, and the bidders whose realized value is below r do not to participate in the auction.*

Hence, by Prop. 6 bidders' expected payoff and seller expected revenue is the same.

Reserves Prices and Entry Fees

Remarks

- Reserve prices and/or entry fees allow the seller to increase its payoff (revenue), but produce inefficient outcomes with positive probability: even though the total surplus is less than the maximum surplus, the distribution of surplus is more favorable to the seller. However, reserve prices/entree fees do not eliminate bidders *information rents*.
- Reserve prices and/or entry fees would destroy the strategic equivalence between SPA and EA, and between FPA and DA, if bidders observed the bidders that effectively participate in the auction and those who do not (because their values are below r).

Readings

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