Midterm – March 2015

Exercise 1. A coastal city is considering building a small artificial beach. It is known that some residents value this public project in $\bar{v} = 2$ monetary units, whereas other residents have no value for it (that is, $\underline{v} = 0$). In order to make a decision, the city council is going to conduct a survey asking each resident whether his/her value is $\bar{v}$. If a majority of residents answer "yes," then the beach will be built and its cost will be shared equally by those who answered "yes" – that is, those who answered "no" will be exempted from paying.

(a) (5 points) Describe the Bayesian game faced by the residents of the city. Assume that the residents values are independent realizations of a random variable that takes values are $\bar{v}$ and $\underline{v}$ with probabilities $q \in (0, 1)$ and $1 - q$, respectively.

(b) (15 points) Assume that there are 3 residents, and that the cost of the project is 3 monetary units. Determine the set of values of q for which sincere voting is a Bayesian Nash equilibrium. Is the sincere BNE efficient?

(c) (10 points) For the values of q for which sincere voting is not a BNE, is there a (mixed strategy) symmetric BNE? In this BNE efficient?

(a) The Bayesian game played by the residents , $\Gamma = (N, T, A, u, p)$, is described by the following ingredients:

- $N = \{1, ..., n\}$
- $T = \{\bar{v}, \underline{v}\}^n$
- $A = \{0, 1\}^n$, *where* 0 *(respectively* 1*) represents answering no (yes)*
- $u_i : A \times T \to R$ *is defined as*

$$u_i(a, t) = \begin{cases} 0 & \text{if } \sum_{j=1}^n a_j < n/2 \\ 0 & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \underline{v} \text{ and } a_1 = 0 \\ -3/\sum_{j=1}^n a_j & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \underline{v} \text{ and } a_1 = 1 \\ 2 & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \bar{v} \text{ and } a_1 = 0 \\ 2 - 3/\sum_{j=1}^n a_j & \text{if } \sum_{j=1}^n a_j \geq n/2,\ t_i = \bar{v} \text{ and } a_1 = 1. \end{cases}$$

- *For each* $t \in T$, $p(t) = \prod_{j=1}^n \rho(t_i)$, *where* $\rho(\bar{v}) = q$ *and* $\rho(\underline{v}) = 1 - q$.

(b) The sincere strategy is $s_i^*(\underline{v}) = 0$ *and* $s_i^*(\bar{v}) = 1$.

An inspection of the function u_i *reveals that when the individual's value is* $\underline{v}$ *answering "yes" is an action (weakly) dominated: in the best case it generates a payoff equal to zero, and in the worst case a negative payoff. Therefore answering "no" (that is, saying the truth,* $s_i^*(\underline{v}) = 0$*) is optimal when* $t_i = \underline{v}$.

If the other residents answer sincerely, then the expected payoff of a resident i *whose value is* $t_i = \bar{v}$ *and answers (sincerely) "yes"* ($a_i = 1$)

$$U_i(s_{-i}^*, 1/\bar{v}) = q^2\,(2-1) + 2q\,(1-q)\left(2 - \frac{3}{2}\right) = q,$$

whereas her expected utility when she lies (that is, takes action $a_i = 0$*, which is to answer "no") is*

$$U_i(s_{-i}^*, 0/\bar{v}) = 2q^2.$$

In order for sincere behavior to be a BNE we need $q \geq 2q^2$*, that is,* $q \leq 1/2$.

The sincere BNE are efficient because the beach is built only when its social value ($t_1 + t_2 + t_3$) *is greater than its cost (3), since only in this case there are at least two individual whose value is* $\bar{v}$ *and therefore respond "yes."*

(c) Assume that $q > 1/2$*. Since answering "yes"* ($a_i = 1$) *is (weakly) dominated the individual's value is* $\underline{v}$*, let us consider the mixed strategy profile* σ^* *given for each* $i \in N$*, by* $\sigma_i^*(1/\underline{v}) = 0$ *and* $\sigma_i^*(1/\bar{v}) = \alpha \in (0,1)$*. We have*

$$U_i(\sigma_{-i}^*, 1/\bar{v}) = q^2\alpha^2\,(2-1) + q^2\,(2\alpha\,(1-\alpha))\left(2 - \frac{3}{2}\right) + 2q\,(1-q)\,\alpha\left(2 - \frac{3}{2}\right) = q\alpha,$$

and

$$U_i(\sigma_{-i}^*, 0/\bar{v}) = 2q^2\alpha^2.$$

In order for σ^* *to be a BNE we need that* $U_i(\sigma_{-i}^*, 1/\bar{v}) = U_i(\sigma_{-i}^*, 0/\bar{v})$*, that is,*

$$q\alpha = 2q^2\alpha^2.$$

Hence $\alpha^* = 1/2q$*. Since* $q > 1/2$*, we have* $\alpha^* < 1$*. Therefore when sincere behavior is not a BNE there is a symmetric BNE in mixed strategies.*

These BNE in mixed strategies are inefficient since there is a positive probability that beach is not built despite the fact that its social value is greater than its cost.

Exercise 2. Consider the contract design problem of a Principal whose revenue is a random variable taking values $x_1 = 4$ and $x_2 = 18$ with probabilities that depend on whether or not the Agent exerts effort, $e \in \{0, 1\}$; specifically, $p_1(0) = p_2(0) = 1/2$, whereas $p_1(1) = 1/4$ and $p_2(1) = 3/4$. The Agent's reservation utility is $\underline{u} = 1$, and his cost of effort is $v(1) = 1 > 0 = v(0)$.

(a) (20 points) Assuming that the Principal is risk-neutral and the preferences of the Agent are represented by the von Neumann-Morgenstern utility function $u(x) = \sqrt{x}$, determine the optimal contract when effort is verifiable.

(b) (10 points) Under the assumptions in (a), determine the optimal contract when effort is *not* verifiable.

(c) (10 points) Assuming that the Principal is risk averse and the Agent is risk-neutral, determine the optimal contract when effort is verifiable and when it is *not* verifiable.

(a) The optimal contract not requiring the Agent exerting effort, i.e., $e = 0$*, involves paying the Agent a fixed wage* $\bar{w}(0) = 1$*, which is obtained solving the participation constraint with equality,*

$$Eu(\bar{w}(0)) = \underline{u} + v(0) \Leftrightarrow \sqrt{\bar{w}(0)} = 1 + 0.$$

Profit is $y(0) = E(X(0) - 1) = 10$.

The optimal contract requiring the Agent exerting effort (i.e., $e = 1$*) involves paying the Agent a fixed wage* $\bar{w}(1) = 4$*, which is obtained solving the participation constraint with equality,*

$$Eu(\bar{w}(1)) = \underline{u} + v(1) \Leftrightarrow \sqrt{\bar{w}(1)} = 1 + 1.$$

Profit is $y(1) = E\left(X(1)) - 4\right) = 10.5$.

Hence when effort is verifiable the optimal contract is $(e^*, \bar{w}(e^*)) = (1, 4)$.

(b) If effort is not verifiable, then the contract $(0, \bar{w}(0))$ *continues to satisfy the participation and incentive constraints. However, if the Principal wants the Agent to exert effort, then the wage contract* $W^*(1) = (w_1(1), w_2(1))$ *must satisfy the participation and incentive constraint with equality, that is,*

$$\begin{aligned} \frac{1}{4}\sqrt{w_1(1)} + \frac{3}{4}\sqrt{w_2(1)} &= 2 \\ \frac{1}{4}\sqrt{w_1(1)} + \frac{3}{4}\sqrt{w_2(1)} - 1 &= \frac{1}{2}\sqrt{w_1(1)} + \frac{1}{2}\sqrt{w_2(1)} \end{aligned}$$

Unfortunately, this system does not have a real solution. (I am sorry, I did not plan this – it is just a mistake.)

If we assume that the minimum wage the Principal can pay is $w = 0$*, then the most favorable contract is* $w_1(1) = 0$ *and*

$$\begin{aligned} \frac{3}{4}\sqrt{w_2(1)} &\geq 2 \\ \frac{1}{4}\sqrt{w_2(1)} &\geq 1, \end{aligned}$$

that is, $w_2(1) = 16$*. The Principal's profit with this contract is*

$$E\left(X(1)\right) - E(W^*(1)) = \left(\frac{1}{4}\right)(4) + \left(\frac{3}{4}\right)(18 - 16) = \frac{5}{2} < 10.$$

Hence the optimal contract is $(e^*; W(e^*)) = (0; 1, 1)$*.*

(c) In this case the optimal contract involves a franchise, i.e., a fixed payment to the Principal y^* *given by*

$$y^* = \max_{e \in \{0,1\}} E(X(e)) - v(e) - \underline{u}.$$

If we normalize the Agent's utility function to be $u(x) = x$*, and maintain his reservation utility at the level* $\underline{u} = 1$*, then our calculations in part (a) yield* $y^* = 12.5$*, corresponding to optimal level of effort* $e^* = 1$*.*

Exercise 3. Consider a *competitive* market for lemons in which there is only one seller who owns one unit of the good and one buyer who wants to buy a single unit of the good. The quality of the good is a random variable X distributed uniformly on $[0,1]$. The realization of X is only observed by the seller. The buyer is risk neutral and her value of the good is $x+\theta$, where x is the realized quality and $\theta \in (0,1/2)$. The seller's opportunity cost is equal to the good's quality, x.

(a) (10 points) Calculate and graph the market demand. (Hints: For which realization of X will the seller supply at each price $p \in [0,1]$? What is the expected value to the buyer of the qualities that would be supplied at $p \in [0,1]$?)

(b) (10 points) Determine the set of realizations of X for which the competitive equilibrium involves trade.

(c) (10 points) Calculate the expected surplus generated is this market. (Hints: what is the surplus realized when there is trade? What is the probability that the buyer and the seller trade?) Calculate the effect on the expected surplus of a unit subsidy $s \in [0,1]$ to the buyer (which the receives only if she buys the good.) (Hints: How does a subsidy s affects the value of the buyer? How does it affect the probability of trade?) Calculate the smallest subsidy that maximizes the expected surplus.

(a) For each realization x of X the supply is

$$S(p,x) = \begin{cases} 0 & \text{if } p < x \\ \{0,1\} & \text{if } p = x \\ 1 & \text{if } p > x. \end{cases}$$

Since at each price p only the qualities $x \leq p$ are supplied, the expected quality of unit supplied at p is $\mathbb{E}(X|\,X \leq p) = p/2$. (Although this is clear by noticing that $X|\,X \leq p$ is uniformly distributed in $[0,p]$, this expectation may be calculated formally.(*)) Hence the buyer demands a unit of the good when

$$p < \mathbb{E}(X|\,X \leq p) + \theta = \frac{p}{2} + \theta \Leftrightarrow p \leq 2\theta,$$

and demands zero if $p > 2\theta$. (If $p = 2\theta$ the buyer is indifferent between buying or not.) The demand is

$$D(p) = \begin{cases} 0 & \text{if } p > 2\theta \\ \{0,1\} & \text{if } p = 2\theta \\ 1 & \text{if } p < 2\theta. \end{cases}$$

(*) In order to calculate $E(X|\,X \leq p)$ formally, one has to obtained the density function of the random variable $X|\,X \leq p$. To do this, note that the cumulative density function of $X|\,X \leq p$ is

$$F_{X|X\leq p}(x) = \Pr(X \leq x|\,X \leq p) = \frac{\Pr(X \leq x, X \leq p)}{\Pr(X \leq p)} = \frac{F(x)}{F(p)} = \frac{x}{p}$$

if $x \leq p$, and $F_{X|X\leq p}(x) = 0$ if $x > p$. The density function of $X|\,X \leq p$ is $f_{X|X\leq p}(x) = F'_{X|X\leq p}(x) = 1/p$ if $x \leq p$ and $f_{X|X\leq p}(x) = 0$ if $x > p$. therefore

$$\mathbb{E}(X|\,X \leq p) = \int_0^p x f_{X|X\leq p}(x)dx = \int_0^p \frac{x}{p}dx = \frac{p}{2}.$$

(b) Suppose that $x < 2\theta$. Then

$$D(p) = 1 > 0 = S(p, x)$$

for $p \in (0, x)$,

$$D(p) = 1 = S(p, x)$$

for $p \in (x, 2\theta)$, and

$$D(p) = 0 < 1 = S(p, x)$$

for $p \in (2\theta, 1)$. Thus, in the competitive equilibrium there is trade at a price $p^* \in [x, 2\theta]$.

Suppose that $x > 2\theta$. Then $D(p) = 0$ for all $p \in (x, 1)$, and $S(p) = 0$ for $p \in (0, x)$. Hence there is not trade at the competitive equilibrium.

(c) The surplus when there is trade is the difference between the buyer's value and the seller's cost:

$$S(x) = (x + \theta) - x = \theta.$$

The probability that there is trade is

$$\Pr(X \leq 2\theta) = \int_0^{2\theta} f(x)dx = \int_0^{2\theta} dx = 2\theta.$$

Hence the expected surplus is

$$\bar{S}(X) = \theta \Pr(X \leq 2\theta) = 2\theta^2.$$

With a subsidy s the value of the good to the buyer is $x + \theta + s$, and the demand is

$$D(p) = \begin{cases} 0 & \text{if } p > 2(\theta + s) \\ \{0, 1\} & \text{if } p > 2(\theta + s) \\ 1 & \text{if } p < 2(\theta + s) \end{cases}$$

Thus, there is trade with a probability

$$\Pr(X \leq 2(\theta + s)) = \int_0^{2(\theta+s)} dx = 2(\theta + s)$$

if $s < 1/2 - \theta$, and $\Pr(X \leq 2(\theta + s)) = 1$ if $s \geq 1/2 - \theta$. The maximum surplus is realized when there is always trade; that is, when $\Pr(X \leq 2(\theta + s)) = 1$. Hence the minimum subsidy that guarantees that there is trade is

$$s^* = 1/2 - \theta.$$