Theory of the Firm

The Firm's Problem: Costs and Profits

Firm's Problem: Description

- We consider a firm producing a single good Q using two inputs: L (labour) and K (capital).
- The technology of the firm is described by the production function, F(L,K), which provides the maximum level of output that can be obtained for each input combination, (L,K).
- Let p be the market price of good Q, w the price of the labour input (L) and r the price of the capital input (K).
- We assume that the firm is competitive in the input markets; i.e., (w,r) are given.

Firm's Problem: Description

- As the goal of the firm is profit maximization (profit = revenue – cost), we can write the firm's problem in the following way:

$$
\begin{aligned}
&Max\ pQ - wL - rK \\
&s.t.\quad Q \leq F(L, K) \\
&\qquad\quad Q \geq 0, L \geq 0, K \geq 0
\end{aligned}
$$

- The decision variables are: Q, L, K, p?

Firm’s Problem: Description

- In order to determine whether or not p is a decision variable, or whether it depends on Q, etc., we need information about the product market:

  - Is it competitive? If it is, p is given (is considered like “data” by the firm).

  - Does the firm have some market power? If it does, p is not independent of Q.

Cost Minimization

- Now, let us postpone the problem of profit maximization and let us think of the “internal” problem of the firm taking the production level as given: Q_0 .

- Given Q_0 , the goal of profit maximization implies, as an intermediate goal, the cost minimization of producing Q_0.

Cost Minimization

$$Max_{\{L,K\}}\ pQ_0 - wL - rK,\ being\ pQ_0\ a\ cte$$

$$\Leftrightarrow$$

$$Max_{\{L,K\}}\ -(wL + rK)$$

$$\Leftrightarrow$$

$$Min_{\{L,K\}}\ wL + rK$$

Cost Minimization

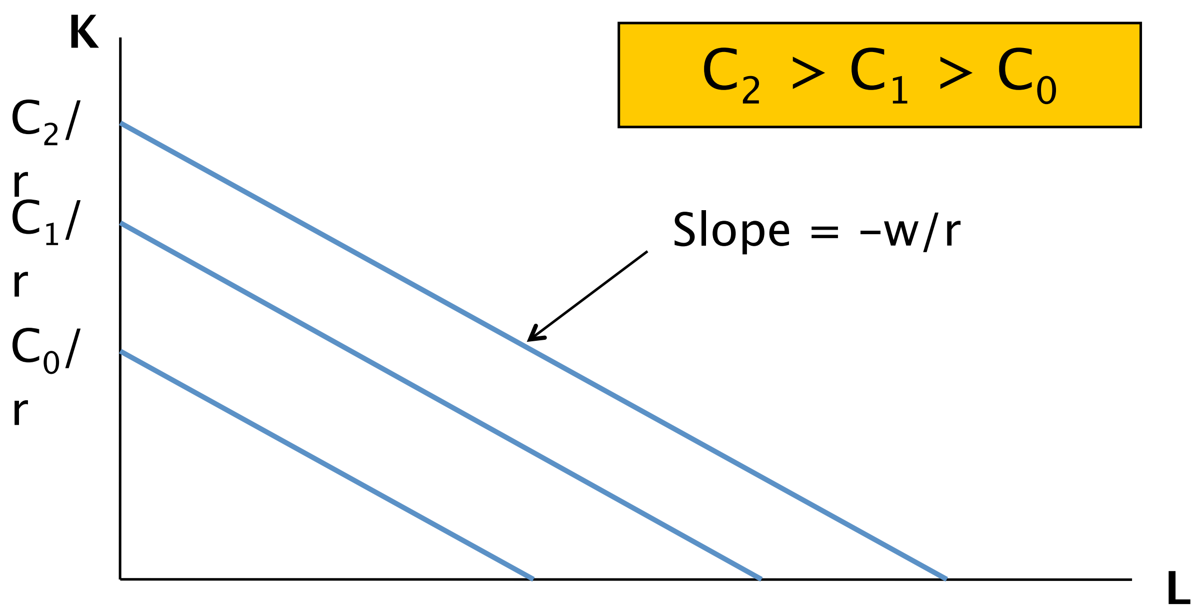
- Given Q_0, profit maximization requires cost minimization; that is, the firm's problem is:

$$
\begin{aligned}
&Min_{\{L,K\}} wL + rK \\
&s.t.\ Q \geq F(L,K) \\
&\quad L \geq 0, K \geq 0
\end{aligned}
$$

Cost Minimization

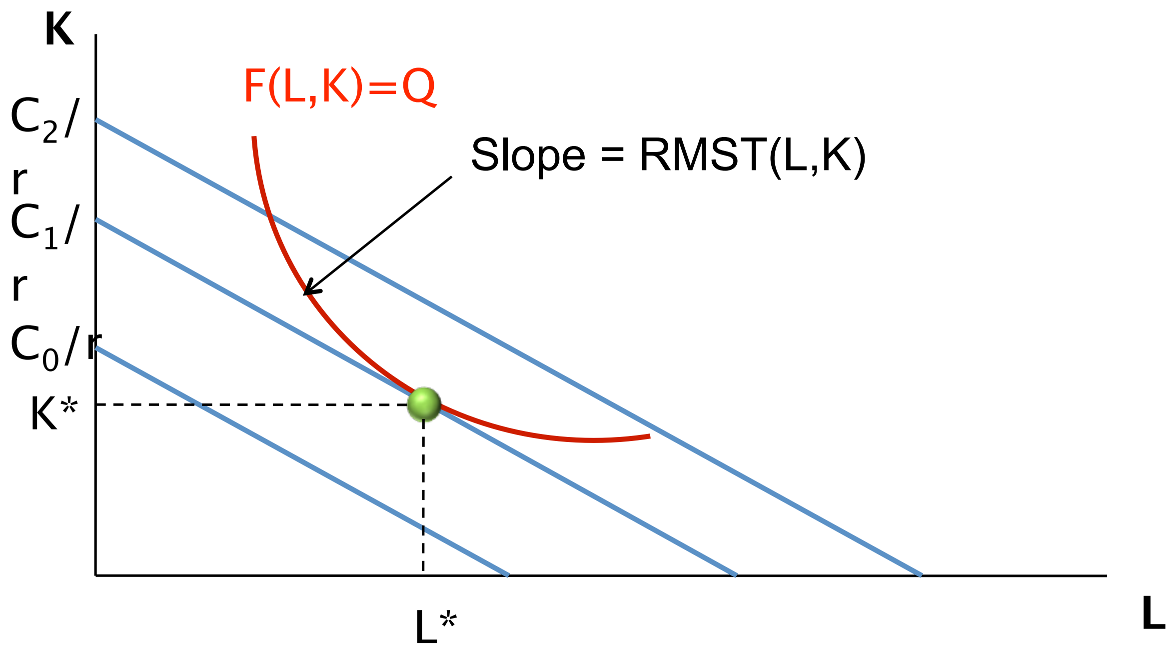
- If C is any cost level, the isocost line $C = wL + rK$ contains all inputs combinations (L,K) which cost C euros. The equation of this straight line is:

$$K = (C/r) - (w/r)L$$



Cost Minimization

- Graphically, the solution of the cost minimization problem is:



Cost Minimization

- The solution of the cost minimization problem is the conditional factor demands:

$$L^* = L(Q,w,r); \quad K^* = K(Q,w,r)$$

- The Total Cost Function provides the minimum cost of producing Q at prices (w,r):

$$C(Q,w,r) = wL^* + rK^* = wL(Q,w,r) + rK(Q,w,r)$$

- Average Total Cost and Marginal Cost are:

$$ATC = C(Q,w,r)/Q$$

$$MC = dC(Q,w,r)/dQ$$

Cost Minimization: Short Run

- In the short run, some of the factors are fixed. Let us suppose in our context that $K=K_0$. Thus, the cost minimization problem is:

$$
\begin{aligned}
&Min_{\{L\}}\ wL + rK_0 \\
&s.t. \quad Q \leq F(L, K_0) \\
&\qquad\quad L \geq 0
\end{aligned}
$$

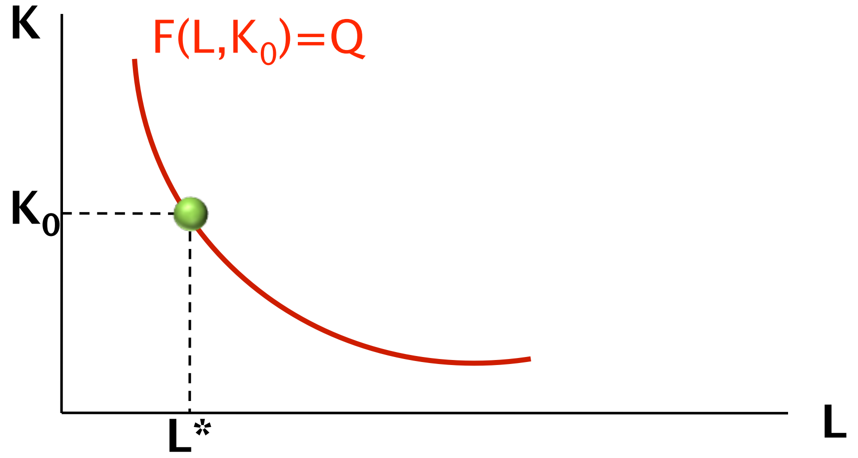
where rK_0 is a constant, which will be named Fixed Cost (FC).

Cost Minimization: Short Run

- With only two inputs, the cost minimization problem in the short run is trivial:

$$F(L,K_0)=Q \rightarrow L^*= L(Q)$$

- In this case, L^* is independent of (w,r), because given K_0 the isoquant determines the conditional labour demand.



Cost Minimization: Short Run

- But in general, with more than two inputs, the cost minimization problem in the short run also makes sense.

- Let us suppose $L = (L_1, L_2)$; $K_0 = (K_{01}, K_{02})$, where $K_0 = (K_{01}, K_{02})$ is a vector of constants.

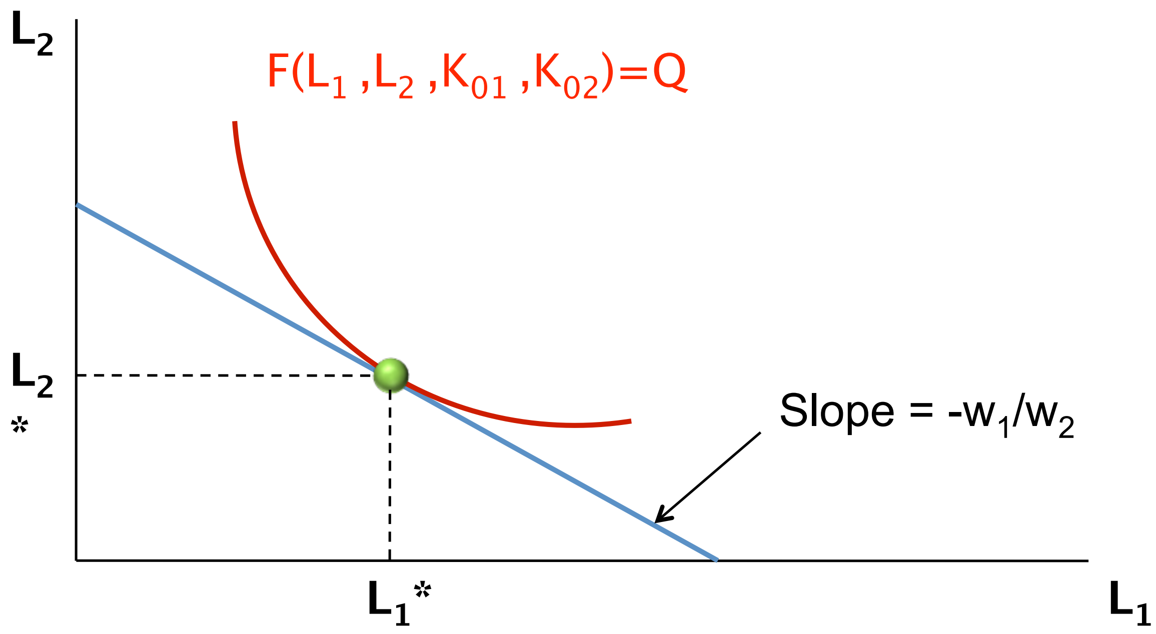
- The problem in this case will be:

$$
\begin{aligned}
& Min_{\{L_1, L_2\}} \; w_1 L_1 + w_2 L_2 + r_1 K_{01} + r_2 K_{02} \\
& s.t. \qquad Q \le F(L_1, L_2, K_{01}, K_{02}) \\
& \qquad\qquad L_1 \ge 0, L_2 \ge 0
\end{aligned}
$$

Cost Minimization: Short Run

- The solution is the conditional factors demands in the short run:

$$L_1^* = L_1(Q, w_1, w_2, r_1, r_2); \quad L_2^* = L_2(Q, w_1, w_2, r_1, r_2)$$



Cost Minimization: Short Run

- Let us go back to the two-inputs case, with only one of them variable in the short run.

- The Total Cost Function in the short run is:

$$CT_{SR}(Q,w,r) = wL(Q) + rK_0 \ ,$$

where wL(Q) is the variable cost in the short run (VC_{SR}), and rK_0 is the fixed cost in the SR (FC_{SR}).

- Average Total Cost in the SR:

$$ATC_{SR} = TC_{SR}(Q,w,r)/Q = AVC_{SR} + AFC_{SR} \ ,$$

where wL(Q)/Q is the average variable cost in the short run, and rK_0 /Q is the average fixed cost in the short run..

Cost Minimization: Short Run

- Marginal Cost in the SR:

$$MC_{SR} = dTC_{SR}(Q)/dQ = dVC_{SR}(Q)/dQ$$

- Remark: the TC_{SR} is different from the TC in the long run, and therefore ATC is also different. MC_{SR} does not coincide either with MC in the long run.

Examples: Costs and Returns to Scale

- Let us suppose these three production functions:

$$(a)\, F(L,K) = LK \text{ ® increasing returns}$$

$$(b)\, F(L,K) = \sqrt{LK} \text{ ® constant returns}$$

$$(c)\, F(L,K) = \sqrt[3]{LK} \text{ ® decreasing returns}$$

- For all of them, the cost minimization problem is:

$$\begin{aligned} &Min_{\{L,K\}} \quad wL + rK \\ &s.t. \qquad Q \leq F(L,K) \\ &\qquad\qquad L \geq 0, K \geq 0 \end{aligned}$$

Examples: Costs and Returns to Scale

- In all these three cases we have interior solutions, so solving the problem of the firm is the same as solving the following system:

$$(1)\quad |MRTS(L,K)| = w/r$$

$$(2)\quad F(L,K) = Q$$

- MRTS coincides for the three functions. Therefore, condition (1) coincides for all of them:

$$K/L = w/r \rightarrow K = (w/r)L$$

Examples: Costs and Returns to Scale

- Let us calculate the solution for the function (a). Plug K = (w/r)L into the production function:

$$Q = KL = (w/r)L \cdot L = (w/r)L^2$$

- Conditional demands for production factors:

$$L^* = \sqrt{\frac{r}{w}Q}\,;\ \ K^* = \sqrt{\frac{w}{r}Q}$$

- Costs:

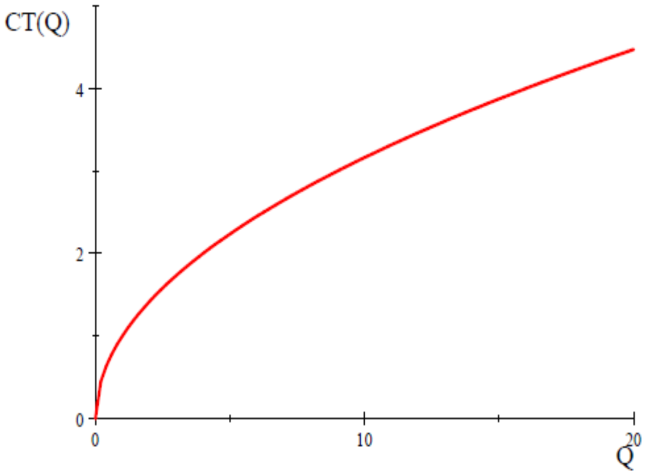
$$TC(Q) = wL^* + rK^* = 2\sqrt{wr}\sqrt{Q}$$

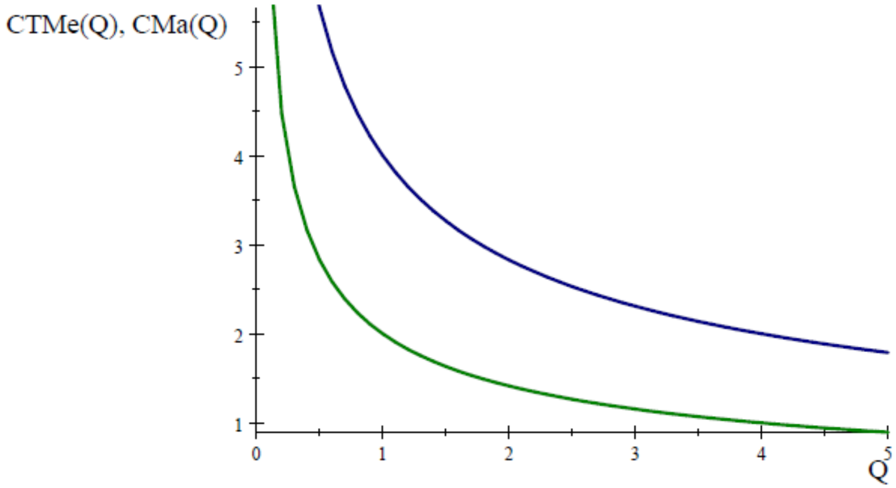
$$ATC(Q) = 2\sqrt{wr}\,\frac{1}{\sqrt{Q}}\,;\ MC(Q) = \sqrt{wr}\,\frac{1}{\sqrt{Q}}$$

Examples: Costs and Returns to Scale

- For w=1, r=4 we get

$$TC(Q) = 4\sqrt{Q};\ ATC(Q) = 4/\sqrt{Q};\ MC(Q) = 2/\sqrt{Q}$$





Examples: Costs and Returns to Scale

- Let us calculate the solution for the function (b). Plug K = (w/r)L into the production function:

$$Q = \sqrt{LK} = \sqrt{(w/r)L^2}$$

- Conditional demands for production factors:

$$L^* = Q\sqrt{\frac{r}{w}};\ K^* = Q\sqrt{\frac{w}{r}}$$

- Costs:

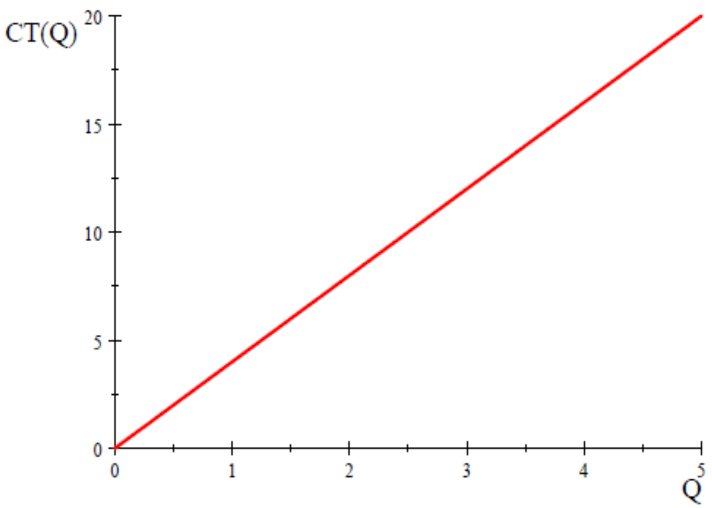
$$TC(Q) = wL^* + rK^* = Q2\sqrt{wr}$$

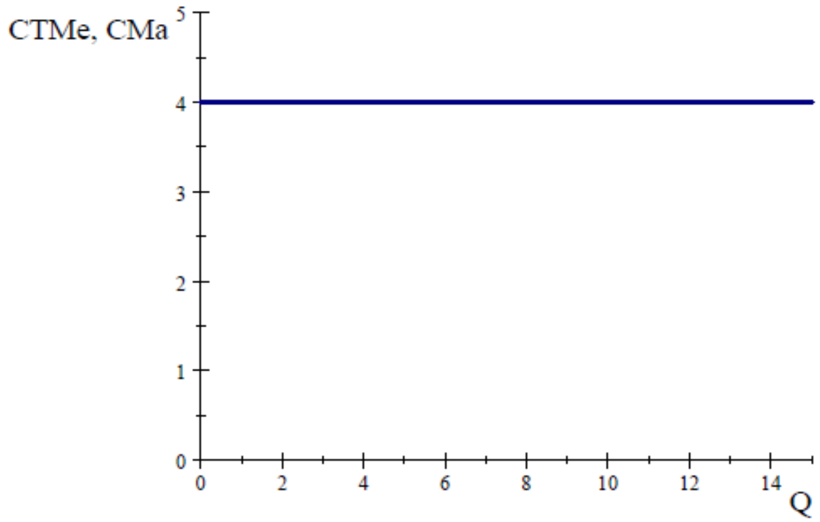
$$ATC(Q) = 2\sqrt{wr};\ MC(Q) = 2\sqrt{wr}$$

Examples: Costs and Returns to Scale

- For w=1, r=4 we get

$$TC(Q) = 4Q;\ ATC(Q) = 4;\ MC(Q) = 4$$





Examples: Costs and Returns to Scale

- Let us calculate the solution for the function (c). Plug K = (w/r)L into the production function:

$$Q = \sqrt[3]{LK} = \sqrt[3]{(w/r)L^2}$$

- Conditional demands for production factors:

$$L^* = Q^{3/2}\sqrt{\frac{r}{w}}; \; K^* = Q^{3/2}\sqrt{\frac{w}{r}}$$

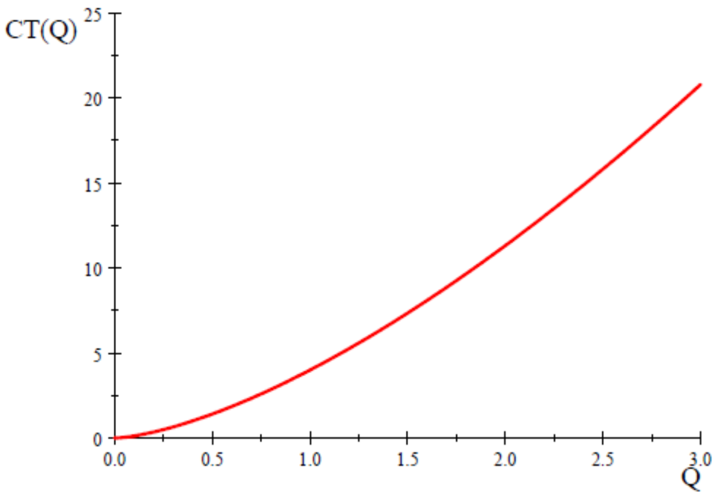
- Costs: $TC(Q) = wL^* + rK^* = Q^{3/2}2\sqrt{wr}$

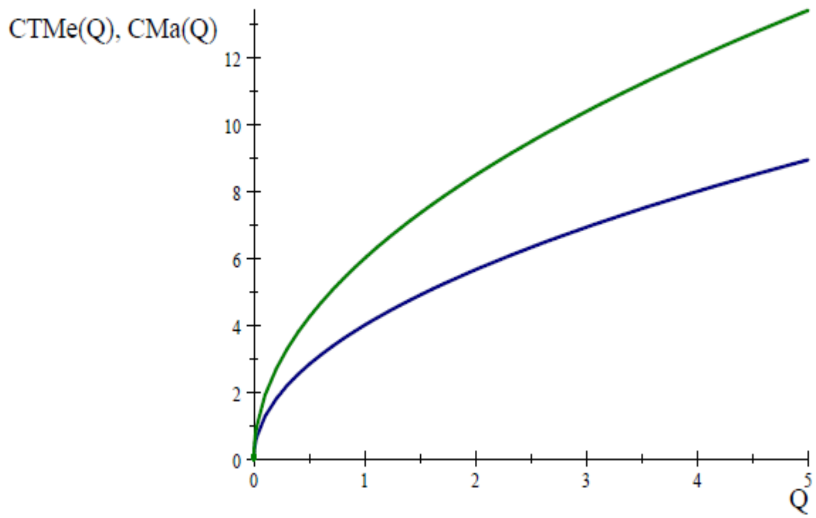
$$ATC(Q) = Q^{1/2}2\sqrt{wr}; \; MC(Q) = Q^{1/2}3\sqrt{wr}$$

Examples: Costs and Returns to Scale

- For w=1, r=4 we get

$$TC(Q) = 4Q^{3/2}; ATC(Q) = 4Q^{1/2}; MC(Q) = 6Q^{1/2}$$





Reconsidering the Firm's Problem

The firm's problem, in the short run or in the long run, can be written as

$$\max \pi(Q) = TR(Q) - TC(Q)$$
$$s.t. \quad Q \geq 0$$

We find the solution using the FOC, and then check the SOC and the Shutting Down Condition:

$$FOC: \quad MR(Q) = MC(Q) \Rightarrow Q^*$$
$$SOC: \quad \pi''(Q) = MR'(Q) - MC'(Q) \leq 0$$
$$SDC: \quad \pi(Q^*) \geq \pi(0)$$

Reconsidering the Firm's Problem

Conditions FOC, SOC, SDC provide a solution to the firm's problem whether it is a competitive firm or not -- for example, when it is a monopoly.

For a competitive firm, the marginal revenue is a constant (the market price). In the monopoly case, however, the marginal revenue will depend on Q.