Final Exam (May 26, 2021)

Answer exercises 1 and 2, and any one of the remaining exercises.

Exercise 1. A pure exchange economy extends over two periods, today and tomorrow. The state of nature tomorrow can either be sunny (S) or cloudy (C). The economy is populated by two consumers, A and B, whose preferences over consumption today (x), tomorrow when sunny (y), and tomorrow when cloudy (z) are represented by a utility function $u_i(x,y,z) = x^{\alpha_i}yz$, where $(\alpha_A, \alpha_B) = (2,1)$, and whose endowments are $(\bar{x}_A, \bar{y}_A, \bar{z}_A) = (\bar{x}_B, \bar{y}_B, \bar{z}_B) = (2,2,2)$.

(a) (25 points) Calculate the competitive equilibrium prices and allocation assuming that there are contingent markets for all goods. (You may find it useful to normalize prices so that $p_x + p_y + p_z = 24$.)

For $i \in \{A, B\}$, consumer i's problem is

$$\max_{(x,y,z)\in\mathbb{R}^3_+} x^{\alpha_i}yz$$
$$\text{subject to: } x + p_y y + p_z z \leq \bar{x}_i + p_y\bar{y}_i + p_z\bar{z}_i.$$

Calculating $MRS_{x,y}(x,y,z) = \alpha_i y/x$ and $MRS_{x,z}(x,y,z) = \alpha_i z/x$, and noting that

$$p_x\bar{x}_i + p_y\bar{y}_i + p_z\bar{z}_i = 2\,(p_x + p_y + p_z) = 48$$

and that $(p_x, p_y, p_z) \gg 0$ in the CE (because u_i is strictly increasing in x, y, and z), consumer i's demands solve the system

$$\begin{aligned} \frac{\alpha_i y}{x} &= \frac{p_x}{p_y} \\ \frac{\alpha_i z}{x} &= \frac{p_x}{p_z} \\ p_x x + p_y y + p_z z &= 48. \end{aligned}$$

Thus,

$$x_i(p) = \frac{48\alpha_i}{(2+\alpha_i)\,p_x},\ y_i(p) = \frac{48}{(2+\alpha_i)\,p_y},\ z_i(p) = \frac{48}{(2+\alpha_i)\,p_z},$$

Hence, market clearing conditions,

$$y_A(p) + y_B(p) = 4,\ z_A(p) + z_B(p) = 4,$$

yield the equations

$$\frac{48}{4p_y} + \frac{48}{3p_y} = 4,\ \frac{48}{4p_z} + \frac{48}{3p_z} = 4.$$

Solving this system of equations we get $p_y^* = p_z^* = 7$, and therefore $p_x^* = 10$. The equilibrium allocation is

$$(x_A^*, y_A^*, z_A^*) = (\frac{24}{10}, \frac{12}{7}, \frac{12}{7}),\ (x_B^*, y_B^*, z_B^*) = (\frac{16}{10}, \frac{16}{7}, \frac{16}{7}).$$

(b) (10 points) Now suppose that there are no contingent markets, but there is a credit market and a market for a security that pays 2 units of consumption tomorrow if sunny and nothing if cloudy. Determine the competitive equilibrium interest rate r^* and security price q^*. (Hint. Normalizing the spot prices to $(\hat{p}_x, \hat{p}_y, \hat{p}_z) = (1, 1, 1)$, you can consolidate an agent's budget constraints into a single equation involving her consumption of goods – you can do this by calculating and substituting the amount she borrows and the number of units of the security she buys as functions of her consumptions. You will see then the relation between (q^*, r^*) and the prices you obtained in part (a).)

For (r, q), the problem of consumer is $i \in \{A, B\}$

$$\begin{aligned} x + qs &\leq 2 + b \\ y &\leq 2 - (1+r)b + 2s \\ z &\leq 2 - (1+r)b. \end{aligned}$$

Clearly, the budget constraints are binding at the solution. Hence, using the last to equations to solve for b and s we get ,

$$\begin{aligned} b &= -\frac{z-2}{1+r} \\ s &= \frac{y-z}{2}. \end{aligned}$$

Substituting b and s in the first equation we get

$$x + q\frac{y-z}{2} + \frac{z-2}{1+r} = 2,$$

i.e.,

$$x + \frac{q}{2}y + \left(\frac{1}{1+r} - \frac{q}{2}\right) z = 2\left(1 + \frac{1}{1+r}\right),$$

which we may write as

$$24\left(\frac{1+r}{2+r}\right) x + 12q\left(\frac{1+r}{2+r}\right) y + 12\left(\frac{1+r}{2+r}\right)\left(\frac{2}{1+r} - q\right) z = 48,$$

The equilibrium security price and interest rate must solve the system

$$\begin{aligned} 24\left(\frac{1+r}{2+r}\right) &= p_x^* = 10 \\ 12\left(\frac{1+r}{2+r}\right) q &= p_y^* = 7 \\ 12\left(\frac{1+r}{2+r}\right)\left(\frac{2}{1+r} - q\right) &= p_z^* = 7. \end{aligned}$$

Using any two of this equations we get

$$(q^*, r^*) = \left(\frac{7}{5}, -\frac{2}{7}\right).$$

And of course, the resulting allocation is that of part (a).

Exercise 2. The revenue of a risk-neutral principal is a random variable X taking values $x_1 = 2$, $x_2 = 4$ and $x_3 = 8$ with probabilities that depends on the level of effort of an agent, $e \in [0, 1]$, and are given by $p_1(e) = (1 - \sqrt{e})/2$ and $p_2(e) = p_3(e) = (1 + \sqrt{e})/4$. Effort is verifiable. There are two types of agents, L and H, present in the population of agents in fractions $q \in (0, 1)$ and $1-q$, respectively. All agents have the same preferences, represented by the Bernoulli utility function $u(w) = w$, and reservation utility $\underline{u} = 0$, but different costs of effort, given by $c_L(e) = c(e)$ and $c_H(e) = kc(e)$, where $c(e) = e$ and $k = 2$.
(a) (10 points) Assuming that an agent's type is observable, determine the contract the principal will offer to each type of agent, and the principal's expected profit.

(a) The expected revenue is

$$\mathbb{E}[X(e)] = 2\left(\frac{1-\sqrt{e}}{2}\right) + (4+8)\left(\frac{1+\sqrt{e}}{4}\right) = 4 + 2\sqrt{e}.$$

Optimal wage offers involve fixed wages: $w_L(e) = e$ *and* $w_H(e) = 2e$. *Effort* e_τ *solves*

$$\max_{e\in[0,1]} \mathbb{E}[X(e)] - w_\tau(e).$$

Thus,

$$\frac{1}{\sqrt{e_L}} - 1 = 0, \text{ and } \frac{1}{\sqrt{e_H}} - 2 = 0$$

that is,

$$e_L = 1 \text{ and } e_H = 1/4.$$

Therefore the optimal contracts are $(e_L, \bar{w}_L) = (1, 1)$ *to the Agent of type* L, *and* $(e_H, \bar{w}_H) = (1/4, 1/2)$ *to the Agent of type* H. *Thus, with complete information the expected profit to the Principal is*

$$\begin{aligned} \Pi_{CI}(q) &= q\left(\mathbb{E}[X(e_L)] - \bar{w}_L\right) + (1-q)\left(\mathbb{E}[X(e_H)] - \bar{w}_H\right) \\ &= q\,(4+2-1) + (1-q)\left(4+1-\frac{1}{2}\right) = \frac{1}{2}\,(9+q)\,. \end{aligned}$$

(b) (25 points) Now assume that an agent's type is *private information*. Identify the principal's optimal menu of contracts for each value of q. (Keep in mind that the principal may choose to offer a single contract acceptable only by the low cost type; you need to verify when this contract or a menu of contracts is optimal.)

The Principal may offer the ***single contract*** $(e, w) = (1, 1)$, *which only* L *type agents accept, leading to an expected profit*

$$\Pi_{SC}(q) = q\left(\mathbb{E}\left[X(1)\right] - 1\right) = 5q,$$

Alternatively, the Principal may design a ***menu of contracts*** *that warrants that agents of both types will accept. As shown in class, the optimal menu is identified by the participation constraint of type* H *and the incentive of type* L,

$$\begin{aligned} u(w_H) - c_H(e_H) + \underline{u} &\Leftrightarrow w_H = 2e_H \\ u(w_L) - c_L(e_L) &= u(w_H) - c_L(e_H) \Leftrightarrow w_L - e_L = w_H - e_H \end{aligned}$$

and by the optimality conditions

$$\begin{aligned} \left(\mathbb{E}\left[X(e_H)\right]\right)' &= \frac{c'(e_H)}{u'(w_H)} + \frac{q}{1-q}(k-1)\frac{c'(e_H)}{u'(w_L)} \Leftrightarrow \frac{1}{\sqrt{e_H}} = \frac{2}{1} + \frac{q}{1-q}(2-1)\frac{1}{1} \\ \left(\mathbb{E}\left[X(e_L)\right]\right)' &= \frac{c'(e_L)}{u'(w_L)} \Leftrightarrow \frac{1}{\sqrt{e_L}} = \frac{1}{1}. \end{aligned}$$

(In the exercise description I inadvertently changed the convention followed in the class notes, in which I denote by q *and* $1-q$ *the fractions of the* H *and* L *worker types. Hence, to apply to this exercise the optimality condition of the* H *type derived in the class notes* q *must be replaced by* $1-q$, *and vice versa.)*

The solution to this system is

$$\tilde{e}_L(q) = 1,\ \tilde{w}_L(q) = 1 + \left(\frac{1-q}{2-q}\right)^2,\ \tilde{e}_H(q) = \left(\frac{1-q}{2-q}\right)^2,\ \tilde{w}_H(q) = 2\left(\frac{1-q}{2-q}\right)^2.$$

Note that $\tilde{e}_H(0) = 1/4$ *and* $\tilde{e}_H(q)$ *decreases with* q *on* $(0, 1)$.

The Principal's profit with this menu of contracts is

$$\begin{aligned} \Pi_{MC}(q) &= q\left(\mathbb{E}\left[X(\tilde{e}_L)\right] - \tilde{w}_L\right) + (1-q)\left(\left(\mathbb{E}\left[X(\tilde{e}_H)\right] - \tilde{w}_H\right)\right) \\ &= q\left(4 + 2\sqrt{1} - \left(1 + \left(\frac{1-q}{2-q}\right)^2\right)\right) + (1-q)\left(4 + 2\left(\frac{1-q}{2-q}\right) - 2\left(\frac{1-q}{2-q}\right)^2\right) \\ &= \frac{9-4q}{2-q}. \end{aligned}$$

Since

$$\Pi_{MC}(q) - \Pi_{SC}(q) = \frac{9-4q}{2-q} - 5q = \frac{1}{2-q}\left(5q^2 - 14q + 9\right) > 0$$

for all $q \in [0, 1)$, *the menu of contracts is optima lfor all* q.

Exercise 3. *A*nn, *B*ob, and *C*onrad are planning to lease an apartment together and must choose its quality as measured by the monthly rental $x \in \mathbb{R}_+$, in thousand euros, they pay. Their preferences are described by the utility function $u_i(x, y) = y + \alpha_i \ln x$, where y denotes money available to spend on other goods. They each is endowed with 2 thousand euros, and their preferences parameters are $\alpha_A = 1$, $\alpha_B = 2/3$ and $\alpha_C = 1/3$, respectively.

(a) (15 points) Determine the Pareto optimal monthly rental and the Lindahl allocation.

A Pareto optimal allocation (x, y_1, y_2, y_3) *is a solution to the system:*

$$\begin{aligned} MRS_1(x, y) + MRS_2(x, y) + MRS_3(x, y) &= 1 \\ y_1 + y_2 + y_3 + 6x &= 6 \end{aligned}$$

where

$$MRS_i(x, y) = \frac{\partial u_i/\partial x}{\partial u_i/\partial y} = \frac{\alpha_i}{x}.$$

The first equation becomes $2/x = 1$. *Hence the optimal level of public good is* $x^* = 2$. *Thus, any allocation* (x, y_1, y_2, y_3) *such that* $x = 2$ *and* $y_1 + y_2 + y_3 = 4$ *is Pareto optimal.*

In a Lindahl equilibrium the system of personalized prices must be such that for $i \in \{A, B, C\}$

$$MRS_i(x, y) = \frac{\alpha_i}{x} = p_i$$

must hold for $x = 2$. *Hence* $p_A = 1/2$, $p_B = 1/3$, $p_C = 1/6$. *Incomes after paying the monthly rental according to Lindahl prices are*

$$(y_A^L, y_B^L, y_C^L) = (2 - 2p_A, 2 - 2p_B, 2 - 2p_C) = (1, \frac{4}{3}, \frac{5}{3}).$$

(b) (15 points) Calculate the monthly rental assuming that it results from voluntary contributions. (Warning: an interior equilibrium, that is, one in which all contributions are positive, does not exist.)

The contribution of individual $i \in \{A, B, C\}$, $z_i \in R_+$ *solves the problem*

$$\begin{aligned} &\max_{z_i \in \mathbb{R}_+} y + \alpha_i \ln(z_{-i} + z_i) \\ &\text{subject to: } y + z_i = 2, \end{aligned}$$

where $z_{-i} = \sum_{j \neq i} z_j$. *This problem is equivalent to*

$$\max_{z_i \in \mathbb{R}_+} V(z_i, z_{-i}) = (2 - z_i) + \alpha_i \ln(z_i + z_{-i}).$$

If $\alpha_i > z_{-i}$, *then*

$$\frac{\partial V(0, z_{-i})}{\partial z_i} = -1 + \frac{\alpha_i}{z_i + z_{-i}} > 0,$$

and $z_i^* = \alpha_i - z_{-i} > 0$; *otherwise* $\partial V(0, z_{-i})/\partial z_i \leq 0$ *and* $z_i^* = 0$. *That is*

$$z_i^* = \max\{\alpha_i - z_{-i}, 0\}$$

Hence $z_A^* \geq \alpha_A - (z_B^* + z_C^*)$, *for otherwise Ann would increase her contribution. Then* $z_B^* = z_C^* = 0$, *and therefore* $z_A^* = 1$. *The resulting allocation is*

$$(x^{VC}, y_A^{VC}, y_B^{VC}, y_C^{VC}) = (1, 1, 2, 2).$$

Exercise 4. In a competitive insurance market firms serve a continuum of individuals with preferences represented by the Bernoulli utility function $u(x) = \ln x$ and initial wealth of $W = 2$ euros, who face the risk of suffering a loss $L = 2$ euros with probability p. For a fraction $\lambda \in (0,1)$ of the individuals $p = p_H = 1/2$, whereas for the remaining fraction $p = p_L = 1/3$. Insurance companies cannot tell whether an individual's probability is of one or the other value.
(a) (20 points) Calculate the policies that will be offered in a competitive equilibrium, and identify the values of λ for which such an equilibrium exists.

(a) As argued in class, a competitive equilibrium, when it exists, is separating, and involves offering the policies $(I_H, D_H) = (p_H L, 0) = (1, 0)$, *and* (I_L, D_L) *satisfying*

$$\begin{aligned} I_L &= p_L(L - D_L) \\ u(W - p_H L) &= (1 - p_H)u(W - I_L) + p_H u(W - I_L - D_L). \end{aligned}$$

Substituting $I_L = p_L(1 - D_L)$ *into the second equation, and using the parameter values and utility function given we get*

$$\ln(2-1) = \frac{1}{2}\ln\left(2 - \frac{2 - D_L}{3}\right) + \frac{1}{2}\ln\left(2 - \frac{2 - D_L}{3} - D_L\right),$$

that is, D_L *solves the equation*

$$1 = \left(2 - \frac{2-x}{3}\right)\left(2 - \frac{2-x}{3} - x\right) \Leftrightarrow 2x^2 + 4x - 7 = 0.$$

The solution is

$$\begin{aligned} D_L &= \frac{3}{\sqrt{2}} - 1 \simeq 1.121 \\ I_L &= \frac{2 - D_L}{3} \simeq 0.293. \end{aligned}$$

The expected utility of a low risk agent with this policy is

$$(1 - p_L)\, u(W - I_L) + p_L u(W - I_L - D_L) = \left(1 - \frac{1}{3}\right)\ln(2 - 0.293) + \frac{1}{3}\ln(2 - 0.293 - 1.121) \simeq 0.178.$$

For this menu to be a CE a low risk individual must not prefer the full insurance pooling policy $(\bar{p}(\lambda)L, 0)$, *with*

$$\bar{p}(\lambda) = \lambda\left(\frac{1}{2}\right) + (1-\lambda)\left(\frac{1}{3}\right) = \frac{2+\lambda}{6}.$$

to the policy (I_L, D_L), *that is,*

$$\begin{aligned} \ln(2 - (2)\,\bar{p}(\lambda)) &\leq 0.178 \\ &\Leftrightarrow \\ 2 - \frac{2+\lambda}{3} &\leq e^{0.178} \\ &\Leftrightarrow \\ \lambda &\geq 3\left(2 - e^{0.178}\right) - 2 \simeq 0.415. \end{aligned}$$

(b) (10 points) Suppose that the government puts to a referendum a law requiring that everyone subscribes a full coverage insurance policy. Identify the values of λ for which such a proposal would be approved by a majority of the electorate.

With a mandatory full insurance policy competitive pressure will force insurance firms to offer the policy $(\bar{I}, 0) = (\bar{p}L, 0)$, *which all individuals would subscribe.*

Obviously, if the fraction of high risk individuals form a majority, that is, if $\lambda > 1/2$, *then the referendum will result in the approval of the law requiring that everyone subscribes a full coverage insurance.*

If $\lambda \in (0.415, 0.5)$, *then the fraction of low risk individuals form a majority, and these individuals are better off subscribing the policy offered in the competitive separating equilibrium. Hence the referendum will result in the rejection of the law requiring that everyone subscribes a full coverage insurance.*

For $\lambda \in (0, 0.415)$ *there is no clear reference to consider since a competitive equilibrium does not exist, and hence the theory does not allow to anticipate the result of the referendum.*

Exercise 5. Consider the true value bidding equilibrium of a sealed-bid second price action in which there are 2 bidders whose values are iid according to the cdf with support $[0,1]$ given by $F_X(x) = 2x - x^2$.

(a) (20 points) Calculate the expected gross surplus, the expected seller's revenue, and a bidder's expected payoff.

The gross surplus is $Y_1^{(2)}$, *whose cdf is* $F_{Y_1^{(2)}}(y) = (F_X(y))^2$. *Hence* $f_{Y_1^{(2)}}(y) = 2(2y - y^2)(2-2y)$, *and*

$$\mathbb{E}[Y_1^{(2)}] = \int_0^1 y f_{Y_1^{(2)}}(y)dy = \int_0^1 2y\left(2y - y^2\right)(2-2y)\,dy = \frac{7}{15}.$$

The seller's revenue is $Y_2^{(2)}$, *whose cdf is*

$$F_{Y_2^{(2)}}(y) = 2F_X(y) - F_X(y)^2 = 2\left(2y - y^2\right) - \left(2y - y^2\right)^2.$$

Hence

$$f_{Y_2^{(2)}}(y) = 4(1-y)^3,$$

and

$$\mathbb{E}[Y_2^{(2)}] = \int_0^1 y f_{Y_2^{(2)}}(y)dy = \int_0^1 4y(1-y)^3\,dy = \frac{1}{5}.$$

Thus, a bidder's expected payoff is

$$U(X) = \frac{1}{2}\left(\mathbb{E}[Y_1^{(2)}] - \mathbb{E}[Y_2^{(2)}]\right) = \frac{1}{2}\left(\frac{7}{15} - \frac{1}{5}\right) = \frac{2}{15}.$$

(b) (20 points) If the seller value is 1/4, what would the reserve price that maximizes the seller expected payoff?

If the seller's value is $x_0 = 1/4$, *then the optimal reserve price solves the equation*

$$r = x_0 + \frac{1}{\lambda(r)},$$

where λ *is the hazard rate of* F_X, $\lambda(x) = f_X(y)/(1 - F_X(x))$. *Hence* r *solves*

$$r = \frac{1}{4} + \frac{1 - (2r - r^2)}{2(1-r)} \Leftrightarrow r = \frac{3}{4} - \frac{1}{2}r,$$

that is,

$$r^* = \frac{1}{2}.$$