Exercise List 3 - Microeconomics II
Moral Hazard - Screening- Signaling

Exercise 1

The optimal contract when effort level is observed:

The optimal contract to implement $e = 0$:

- $w_0 = w_{10} = w_{25} = 1$
- $\pi = 8$.

The optimal contract to implement $e = 1$:

- $w_0 = w_{10} = w_{25} = 4$
- $\pi = 10$.

Since $10 > 8$, the principal will offer the optimal contract to implement $e = 1$.

The optimal contract when effort level is unobserved:

The optimal contract to implement $e = 0$: insert the previous result in the new problem of the principal; it satisfies the IC constraint:

- $w_0 = w_{10} = w_{25} = 1$
- $\pi = 8$.

The optimal contract to implement $e = 1$: consider the system of two equations given by the PC and the IC (binding) conditions:

$$\frac{2}{10}\sqrt{w_0} + \frac{4}{10}\sqrt{w_{10}} + \frac{4}{10}\sqrt{w_{25}} - 1 = 1 \quad (\lambda)$$
$$\frac{2}{10}\sqrt{w_0} + \frac{4}{10}\sqrt{w_{10}} + \frac{4}{10}\sqrt{w_{25}} - 1 = \frac{4}{10}\sqrt{w_0} + \frac{4}{10}\sqrt{w_{10}} + \frac{2}{10}\sqrt{w_{25}} \quad (\mu)$$

The Focs are:

$$\sqrt{w_0} = (\lambda - \mu)/2$$
$$\sqrt{w_{10}} = (\lambda)/2$$
$$\sqrt{w_{25}} = (2\lambda - \mu)/4$$

Defining the objective function of the principal only in terms of w_{25} we get the following wage contracts:

- $w_0 = 0$; $w_{10} = 0$; $w_{25} = 25$.
- $\pi = 4$.

Since $8 > 4$, the principal will offer the optimal contract to implement $e = 0$.

Exercise 2

Part a)

The company's contract design problem to implement No Effort:

$$\max_{w_s, w_{ns}} \frac{1}{2}(200 - w_s) + \frac{1}{2}(0 - w_{ns})$$
$$s.t\ \frac{1}{2}w_s + \frac{1}{2}w_{ns} \geq 50$$
$$\frac{1}{2}w_s + \frac{1}{2}w_{ns} \geq pw_s + (1-p)w_{ns} - 20$$

The solution of this problem is the following:

- $w_s + w_{ns} = 100$ (from the PC)
- $\pi = 50$
- $w_{ns} \geq \frac{30-100p}{1-2p}$ (from the IC)

The company's contract design problem to implement Effort:

$$\max_{w_s, w_{ns}} p(200 - w_s) + (1-p)(0 - w_{ns})$$
$$s.t\ pw_s + (1-p)w_{ns} - 20 \geq 50$$
$$pw_s + (1-p)w_{ns} - 20 \geq \frac{1}{2}w_s + \frac{1}{2}w_{ns}$$

The solution of this problem is the following:

- $pw_s + (1-p)w_{ns} = 70$ (from the PC)
- $\pi = 200p - 70$
- $w_{ns} \geq \frac{100p-70}{2p-1}$ (from the IC)

Comparing the expected profit for the principal we have that the optimal contract proposed by the principal is the following:

- The optimal contract for Effort if $p \geq \frac{3}{5}$
- The optimal contract for No Effort if $p < \frac{3}{5}$.

Given the risk-neutrality of the agent, he will always get 50. In this sense, the social optimality is obtained optimizing the principal utility.

Part b)

The optimal contract to implement $e = 0$: insert the result obtained when the effort is observable (with only the PC constraint) in the problem of the principal; it satisfies the IC constraint:

- $w_s = w_{ns} = e^{\frac{5}{2}}$
- $\pi = 100 - e^{\frac{5}{2}}$.

The optimal contract to implement $e = 1$: consider the system of two equations given by the PC and the IC (binding) conditions:

- $w_s = e^{\frac{25}{6}}$; $w_{ns} = e^{\frac{5}{6}}$
- $\pi = 160 - \frac{4}{5}e^{\frac{25}{6}} - \frac{1}{5}e^{\frac{5}{6}}$.

Since $160 - \frac{4}{5}e^{\frac{25}{6}} - \frac{1}{5}e^{\frac{5}{6}} > 100 - e^{\frac{5}{2}}$, the principal will offer the optimal contract to implement Effort.

Exercise 3

Before calculating the insurance policy offered when the precautionary action is contractible and when it is not, we have to define the reservation utility of the agent: with no policy, will the precautionary action be played by the agent? The expected utility for the agent are the following:

$$\text{with action: } \frac{1}{2}\sqrt{100-36} + \frac{1}{2}\sqrt{100} - \frac{4}{10} = 8.6$$
$$\text{with no action: } \frac{3}{4}\sqrt{100-36} + \frac{1}{4}\sqrt{100} = 8.5$$

Since $8.6 > 8.5$ the action is played by the agent with no insurance and his reservation utility is 8.6.

The optimal contract when the action is contractible:

The optimal contract to implement no action by the agent: consider the following problem for the insurance:

$$\max_{I,D} \frac{1}{4}I + \frac{3}{4}(I - (36 - D))$$
$$s.t\ \frac{1}{4}\sqrt{100 - I} + \frac{3}{4}\sqrt{100 - I - D} \geq 8.6$$

The PC constraint is binding (show by contradiction) and the results are the following:

- $\sqrt{100 - I - D} = \sqrt{100 - I} = 8.6$
- $D = 0$; $I = 100 - (8.6)^2$.

The optimal contract to implement action by the agent: consider the following problem for the insurance:

$$\max_{I,D} \frac{1}{2}I + \frac{1}{2}(I - (36 - D))$$
$$s.t\ \frac{1}{2}\sqrt{100 - I} + \frac{1}{2}\sqrt{100 - I - D} - \frac{4}{10} \geq 8.6$$

The PC constraint is binding (show by contradiction) and the results are the following:

- $\sqrt{100 - I - D} = \sqrt{100 - I} = 8.6 + \frac{4}{10}$
- $D = 0$; $I = 100 - (8.6 + \frac{4}{10})^2$.

The optimal contract when the action is not contractible:

The optimal contract to implement no action by the agent: consider the following problem for the insurance:

$$\begin{aligned} \max_{I,D} \; & \frac{1}{4}I + \frac{3}{4}(I - (36 - D)) \\ s.t \; & \frac{1}{4}\sqrt{100 - I} + \frac{3}{4}\sqrt{100 - I - D} \geq 8.6 \\ & \frac{1}{4}\sqrt{100 - I} + \frac{3}{4}\sqrt{100 - I - D} \geq \frac{1}{2}\sqrt{100 - I} + \frac{1}{2}\sqrt{100 - I - D} - \frac{4}{10} \end{aligned}$$

Inserting the previous result in the new problem of the principal, it results to satisfy the IC constraint. Thus we have again:

- $\sqrt{100 - I - D} = \sqrt{100 - I} = 8.6$
- $D = 0$; $I = 100 - (8.6)^2$.

The optimal contract to implement action by the agent: consider the following problem for the insurance:

$$\begin{aligned} \max_{I,D} \; & \frac{1}{2}I + \frac{1}{2}(I - (36 - D)) \\ s.t \; & \frac{1}{2}\sqrt{100 - I} + \frac{1}{2}\sqrt{100 - I - D} - \frac{4}{10} \geq 8.6 \\ & \frac{1}{2}\sqrt{100 - I} + \frac{1}{2}\sqrt{100 - I - D} - \frac{4}{10} \geq \frac{1}{4}\sqrt{100 - I} + \frac{3}{4}\sqrt{100 - I - D} \end{aligned}$$

In all the previous cases, we can get the solutions from the system of two equations given by the two constraints: recall to show (by contradiction or showing that the multipliers are both strictly greater than zero) that they both bind. From the system we get:

- $\sqrt{100 - I} = 8.6 + \frac{12}{10}$
- $\sqrt{100 - I - D} = 8.6 - \frac{4}{10}$
- $I = 100 - (8.6 + \frac{12}{10})^2$
- $D = (8.6 + \frac{12}{10})^2 - (8.6 - \frac{4}{10})^2$.

Exercise 5

Part a)

The average wage w_1 that can be offered by the (competitive) firm to attract all the workers is the following:

$$w_1 = \text{expected productivity}$$
$$w_1 = 1 * \left[\frac{1}{3} * 1 + \frac{1}{3} * 2 + \frac{1}{3} * 3\right] = 2.$$

With $w_1 = 2$ it is not possible to attract the worker of type 3.

Part b)

The average wage w_2 that can be offered to attract workers of types 1 and 2 is the following:

$$w_2 = \text{expected productivity}$$
$$w_2 = 1 * \left[\frac{1}{2} * 1 + \frac{1}{2} * 2\right] = 1.5.$$

With $w_2 =$ ~~2~~ 1.5 it is possible to attract workers of types 1 and 2 since their reservation utilities are lower. Is $w_2 =$ ~~2~~ 1.5 an equilibrium wage? Yes; show by contradiction that setting up a greater and a lower wage than $w_2 =$ ~~2~~ 1.5 is detrimental for the firm (it is a Nash equilibrium).

Exercise 6

Part a)

The demand for the monopolist is the following:

$$\begin{cases} 18 - 2p, & \text{if } p < 8 \\ 10 - p, & \text{if } p \in [8, 10] \\ 0, & \text{if } p > 10 \end{cases}$$

The result of the monopolist's maximization is $p = 5$. The steps to get this results are the following:

- Maximize the profit for $p < 8$ and obtain the optimal profit and price
- Maximize the profit for $p \in [8, 10]$ and obtain the optimal profit and price (be careful: corner solution)
- Check what this the greatest profit between the two.

Part b)

If the monopolist can distinguish the two types of clients, two contracts (T_c, p_c) and (T_f, p_f) will be set up:

- $T_c = \frac{49}{2}$, $p_c = 1$
- $T_f = \frac{81}{2}$, $p_f = 1$.

Part c)

If the monopolist cannot distinguish the two types of clients two strategies can be pursued:

1. Propose a pooled contract for both with the following structure:

$$(T_{pooled}, p_{pooled}) = (\frac{(8 - p_{pooled})^2}{2}, p_{pooled})$$

   The result of the monopolist profit maximization with such a contract is the following:

   - $p_{pooled} = 2$
   - $T_{pooled} = 18$
   - $\pi_{pooled} = 50$.

2. Propose separate contracts for the consumer and the firm such that each client has incentive to choose the contract specifically defined for his type. The contract menu is given by (T_c, p_c) and (T_f, p_f) and the monopolist maximization problem is defined as follows:

$$\max_{T_c, p_c, T_f, p_f} \quad T_c + T_f + (p_c - 1)(8 - p_c) + (p_f - 1)(10 - p_f)$$

$$s.t \; \frac{(8 - p_c)^2}{2} - T_c \geq 0 \qquad (PC_c)$$

$$\frac{(10 - p_f)^2}{2} - T_f \geq 0 \qquad (PC_f)$$

$$\frac{(8 - p_c)^2}{2} - T_c \geq \frac{(8 - p_f)^2}{2} - T_f \qquad (IC_c)$$

$$\frac{(10 - p_f)^2}{2} - T_f \geq \frac{(10 - p_c)^2}{2} - T_c \qquad (IC_f)$$

   To solve the maximization show that:

   (a) (PC_f) is NOT binding: use the other three constraints

   (b) (PC_c) is binding: by contradiction

   (c) (IC_f) is binding: by contradiction

   (d) (IC_c) is NOT binding.

From the two binding conditions we get that:

- $T_c = \frac{(8-p_c)^2}{2}$
- $T_f = \frac{(10-p_f)^2}{2} - \frac{(10-p_c)^2}{2} - T_c = \frac{(10-p_f)^2}{2} - \frac{(10-p_c)^2}{2} - \frac{(8-p_c)^2}{2}$.

Adding the results for T_c and T_{cf} into the monopolist profit, we have a new maximization problem only in terms of p_c, p_f. Solving the new problem, we get the following results:

- $p_c = 3$; $p_f = 1$
- $T_c = 12.5$; $T_f = 28.5$
- $\pi_{separating} = 63$.

Given that $\pi_{separating} = 63 > \pi_{pooling} = 50$, the monopolist will use the menu of contracts.

Exercise 7(1)

TBC

Exercise 7(2)

Part a)

$c_G = (w_G, e_G) = (k^2/4, k/2), c_B = (w_B, e_B) = (k^2/8, k/4)$

Part b,c)

$c_G = (w_G, e_G) = (2e_B^2, k/2), c_B = (w_B, e_B) = (e_B^2 + e_G^2, \frac{(1-q)k}{2(2-q)})$

Exercise 7(3)

Part a)

$$\begin{aligned}
&\max -0.2w_L - 0.8w_N \\
&\text{s.t.} 0.2\sqrt{w_L} + 0.8\sqrt{w_N} \geq 10 \quad (\lambda) \\
&\quad 0.02\sqrt{w_L} + 0.98\sqrt{w_N} \leq 1 \quad (\mu)
\end{aligned}$$

Part b)

$w_N = 0, w_L = 2500$

Part c)

400

Exercise 7(4)

Part a)

see 7(2)

Part b)

$e_G = 0.5, w_G = 0.25, \Pi = 1/8$

Part c)

Prefers a)

Exercise 7(6)

TBC

Exercise 8

A PBNE is a profile (y_L, y_H, w, μ) where $y_L, y_H \in \mathbb{R}_+$ are the workers signals, and w and μ are mappings from indicating for each $y \in \mathbb{R}_+$ the firm's wage offer, $w(y) \in \mathbb{R}$, and probability with which the firm believes the worker is of type H, $\mu(y) \in [0,1]$.

(a) The most efficient pooling (no-signalling) PBNE is:
$y_L = y_H = 0$, $w(y) = \frac{1}{2}x_H + \frac{1}{2}x_L = 8$ and $\mu(y) = 1/2$ for all $y \in \mathbb{R}_+$.

(b) The most efficient separating (signalling) PBNE is: $y_L = 0$, $y_H = 8$,

$$w(y) = \begin{cases} 4 & \text{if } y < 8 \\ 12 & \text{if } y \geq 8 \end{cases}, \mu(y) = \begin{cases} 0 & \text{if } y < 8 \\ 1 & \text{if } y \geq 8 \end{cases}.$$

Note that the utility of a worker of type L who signals $y \geq 8$ is $12 - y \leq 4 - y_L$, and the utility of a worker of type H who signals $y < 8$ is $4 - y/4 \leq 4 < 12 - y_H/4 = 10$.

Obvously, the workers of type H (L) prefer the separating (pooling) equialbrium.

Exercise 9

- 3 tipos de trabajadores: $a_1 = 1$, $a_2 = 2$ y $a_3 = 3$ con utilidades de reserva $u_1^R = 0.5$, $u_2^R = 1$ y $u_3^R = 2.2$. La probabilidad de cada tipo es 1/3.
- los trabajadores pueden educarse (e) y tiene utilidad $u_i = w - \dfrac{e}{a_i^2}$
- una empresa de una industria perfectamente competitiva quiere contratarlos pero no reconoce la habilidad del trabajador. tiene función de producción $y = a_1 L_1 + a_2 L_2 + a_3 L_3$

1. Calcular e_1, e_2, e_3 tales que el trabajador tipo i prefiera educarse en el nivel e_i y la empresa los trate como a tales al conocer su nivel de educación. Encontrar los niveles mejores para los trabajadores. Señalar explícitamente las creencias de la empresa.

   Solución. Buscamos e^1 y e^2 tales que

$$w(e) = \begin{cases} 1 & e < e^1 \\ 2 & e \in [e^1, e^2) \\ 3 & e \geq e^2 \end{cases}$$

   Ya que la utilidad es decreciente en el nivel de educación, en cualquier equilibrio separador el tipo 1 elegirá $e_1 = 0$, el tipo 2 elegirá $e_2 = e^1$ y el tipo 3 elegirá $e_3 = e^2$.

   Los niveles de educación hay que cumplir las restricciones de participación:

$$u_1(1,0) \geq 0.5 \iff 1 \geq 0.5$$

$$u_2(2,e^1) \geq 1 \iff 2 - \frac{e^1}{4} \geq 1 \iff e^1 \leq 4 \tag{1}$$

$$u_3(3,e^2) \geq 2.2 \iff 3 - \frac{e^2}{9} \geq 2.2 \iff e^2 \leq 7.2 \tag{2}$$

   y las restricciones de incentivos:

$$u_1(1,0) \geq u_1(2,e^1) \iff 1 \geq 2 - e^1 \iff e^1 \geq 1 \tag{3}$$

$$u_1(1,0) \geq u_1(3,e^2) \iff 1 \geq 3 - e^2 \iff e^2 \geq 2 \tag{4}$$

$$u_2(2,e^1) \geq u_2(1,0) \iff 2 - \frac{e^1}{4} \geq 1 \iff e^1 \leq 4 \tag{5}$$

$$u_2(2,e^1) \geq u_2(3,e^2) \iff 2 - \frac{e^1}{4} \geq 3 - \frac{e^2}{4} \iff e^2 - e^1 \geq 4 \tag{6}$$

$$u_3(3,e^2) \geq u_3(1,0) \iff 3 - \frac{e^2}{9} \geq 1 \iff e^2 \leq 18 \tag{7}$$

$$u_3(3,e^2) \geq u_3(2,e^1) \iff 3 - \frac{e^2}{9} \geq 2 - \frac{e^1}{9} \iff e^2 - e^1 \leq 9 \tag{8}$$

$$(1) + (3) + (5) \implies 1 \leq e^1 \leq 4$$

$$(2) + (4) + (7) \implies 2 \leq e^2 \leq 7.2$$

$$(6) + (8) \implies 4 \leq e^2 - e^1 \leq 9$$

   Los mejores niveles de educación para los trabajadores (los más eficientes) son

$$e^1_{\min} = 1, \quad e^2_{\min} = 5$$

   Las utilidades son

$$u_1 = 1, \quad u_2 = 2 - \frac{1}{4} = 1.75, \quad u_3 = 3 - \frac{5}{9} = 2.44$$

Las creencias de la empresa son

$$P(tipo\ 1|e < e^1 = 1$$

$$P(tipo\ 2|e^1 \le e < e^2) = 1$$

$$P(tipo\ 3|e \ge e^2) = 1$$

2. Encontrar el equilibrio con $e_1 = e_2 = 0$ y con $e_3 > 0$ con el nivel de educación más conveniente para los trabajadores del tipo 3. Señalar explícitamente las creencias de la empresa. Compara las utilidades con las encontradas en (a).

   Solución. Ya que $e_1 = e_2 = 0$, no puede distinguir a los dos primeros tipos. Así que buscamos

$$w(e) = \begin{cases} \frac{1+2}{2} = 1.5 & e < e^* \\ 3 & e \ge e^* \end{cases}$$

   Restricciones de participación:

$$u_1(1.5, 0) \ge 0.5 \iff 1.5 \ge 0.5$$

$$u_2(1.5, 0) \ge 1 \iff 1.5 \ge 1$$

$$u_3(3, e^*) \ge 2.2 \iff 3 - \frac{e^*}{9} \ge 2.2 \iff e^* \le 7.2$$

   Restricciones de incentivos:

$$u_1(1.5, 0) \ge u_1(3, e^*) \iff 1.5 \ge 3 - e^* \iff e^* \ge 1.5$$

$$u_2(1.5, 0) \ge u_2(3, e^*) \iff 1.5 \ge 3 - \frac{2^*}{4} \iff e^* \ge 6$$

$$u_3(3, e^*) \ge u_3(1.5, 0) \iff 3 - \frac{e^*}{9} \ge 1.5 \iff e^* \le 13.5$$

   Hence, $e^* \in [6, 7.2]$. El nivel de educación más conveniente para los trabajadores del tipo 3 es $e^*_{\text{min}} = 6$. Por lo tanto, las utilidades son

$$u_1 = u_2 = 1.5, \quad u_3 = 3 - \frac{6}{9} = 2.33$$

   Es decir, el tipo 3 es mejor en (a). Las creencias de la empresa son

$$P(tipo\ 1|e < e^*) = P(tipo\ 2|e < e^*) = \frac{1}{2}, \quad P(tipo\ 3|e \ge e^*) = 1$$

Exercise 10

Part a) $\beta = 0$

Pooling:

$y_L = y_H = y^* \in [0, 1/8]$

$$w^*(y) = \begin{cases} 1 & \text{if} \quad y < y^* \\ 1.5 & \text{if} \quad y \geq y^* \end{cases}$$

$y^* = 0$ is the most efficient

Separating:

$y_L = 0, y_H = y^* \in [1/4, 1]$

$$w^*(y) = \begin{cases} 1 & \text{if} \quad y < y^* \\ 2 & \text{if} \quad y \geq y^* \end{cases}$$

$y^* = 1/4$ is the most efficient

Part a) $\beta = 1$

Notice that under perfect information, agents choose non-zero education. In particular they would choose: $\hat{y}_L = 0.015, \hat{y}_H = 0.25$.

Pooling:

There is no pooling eq.

Separating:

$y_L = 0.015, y_H = y^* \in [0.39, 2.25]$

$$w^*(y) = \begin{cases} 1 + \sqrt{y} & \text{if} \quad y < y^* \\ 2 + \sqrt{y} & \text{if} \quad y \geq y^* \end{cases}$$