Final Exam (June 20, 2022)

Exercise 1. (30 points) Consider an economy that extends over two periods, today and tomorrow, in which there is a single perishable good, rice. In this economy there is a credit market which operates today, and there are spot markets operating tomorrow. There is a price-taking firm with a technology that for each unit of today's rice used as input produces 2 units of rice tomorrow if the state of nature is rainy, but only 1 unit if the state of nature is dry. Both states of nature are equally likely. The firm borrows rice today and supplies rice tomorrow with the objective of maximizing expected profits. There are two consumers whose preferences over consumption of rice today (x), tomorrow if rainy (y) and tomorrow if dry (z) are represented by the utility function $u(x, y, z) = x\,(y + z)$. Consumers share equally the property of the firm and each is endowed with 3 units of rice today. Calculate the competitive equilibrium prices and allocation.

Notation. Denote by b the firm's demand of credit, by l a consumer's credit supply, by r the interest rate, and by p_y and p_z the prices of goods y and z, respectively. You may want to normalize one of these prices; for example, you may want to set $p_y = 1$.

Hints. Since the firm has constant returns to scale, in a competitive equilibrium (CE) the interest rate and prices must satisfy a certain equation. Also, if you write a consumer's budget constraints (note that both consumers are identical), you will see that l is the only choice. Solving a consumer's problem and using the market clearing conditions you can easily solve for the equilibrium prices.

Solution. If the firm borrows b units of rice today, then tomorrow it supplies inelastically $2b$ units of rice if the state is rainy and b units of rice if the state is dry. Hence b solves the problem

$$\max_{b\in\mathbb{R}_+} E[\pi(b)] := \frac{1}{2}\,(2b)\,p_y + \frac{1}{2}\,(b)\,p_z - (1+r)\,b = \Big(p_y + \frac{p_z}{2} - (1+r)\Big)\,b.$$

(The firm's expected profit is a linear function because the firm has constant returns to scale.) The firm's demand of credit is

$$b(p_y, p_z, r) = \begin{cases} 0 & \text{if } p_y + \dfrac{p_z}{2} < 1 + r \\ [0, \infty) & \text{if } p_y + \dfrac{p_z}{2} = 1 + r \\ \infty & \text{if } p_y + \dfrac{p_z}{2} > 1 + r. \end{cases}$$

Thus, in a CE $p_y + p_z/2 \leq 1 + r$, and therefore the firm's expected profits are zero.

Since the firm's profits are zero in a CE, each consumer faces the budget constraints

$$x = 3 - l,\ p_y y = (1+r)\,l,\ p_z z = (1+r)\,l,$$

where l *are units of rice the consumer lends. A consumer's supply of credit* l *solves the problem*

$$\max_{l \in \mathbb{R}} (3-l)\left(\frac{(1+r)\,l}{p_y} + \frac{(1+r)\,l}{p_z}\right) = \left(\frac{1+r}{p_y} + \frac{1+r}{p_z}\right)(3-l)\,l.$$

The solution to this problem is

$$3 - 2l = 0 \Leftrightarrow l(p_y, p_z, r) = \frac{3}{2};$$

that is, each consumer supplies $3/2$ *units of rice today as credit irrespective of the prices.*

Market clearing in the credit market requires

$$b(p_y, p_z, r) = 2l(p_y, p_z, r) = 3 > 0,$$

which in turns requires

$$p_y + \frac{p_z}{2} = 1 + r. \tag{1}$$

Also, the market clearing conditions in the spot markets for rice tomorrow are

$$\begin{aligned}
\frac{1+r}{p_y}\left(\frac{3}{2}\right) + \frac{1+r}{p_y}\left(\frac{3}{2}\right) &= 2\,(3) \Leftrightarrow \frac{1+r}{p_y} = 2 \\
\frac{1+r}{p_z}\left(\frac{3}{2}\right) + \frac{1+r}{p_z}\left(\frac{3}{2}\right) &= 3 \Leftrightarrow \frac{1+r}{p_z} = 1
\end{aligned}$$

Hence

$$1 + r = p_z = 2p_y. \tag{2}$$

The CE interest rate and prices must therefore satisfy equations (1) and (2). A particular CE equilibrium price vector is $(p_y^*, p_z^*, r^*) = (1, 2, 1)$. *The CE allocation is*

$$(x_1^*, y_1^*, z_1^*) = (x_2^*, y_2^*, z_2^*) = (\frac{3}{2}, 3, \frac{3}{2}).$$

Exercise 2. The revenue of a risk-neutral principal is a random variable X taking values $x_1 = 0$, $x_2 = 11$ and $x_3 = 25$ with probabilities that depends on the level of effort of an agent $e \in [0,1]$. Specifically, $p_2(e) = p_3(e) = (1+e)/6$. There is a population of agents with preferences represented by the Bernoulli utility function $u(w) = \sqrt{w}$, and reservation utility $\underline{u} = 2$. The costs of effort for half of these agents is $c_L(e) = e$, while for the other half it is $c_H(e) = 2e$.
(a) (10 points) Determine the contract the principal will offer to each type of agent when agents' types are observable and effort is verifiable.
(b) (15 points) Determine the contract the principal will offer to each type of agent when agents' types are observable, but effort is NOT verifiable, and feasible levels are $e \in \{0,1\}$.
(c) (15 points) Determine the menu of contract the principal will offer when agents' types are NOT observable, but effort is verifiable.

Comments. In part (a) you will arrive at the conclusion that the optimal effort for type L is the maximum possible – the Principal's problem has a corner solution in this case. You will encounter a similar issue in part (c) again: the system that identifies an interior solution to the menu design problem involves a negative effort for a type H agent and an effort above 2 for a type L agent. You should set the effort for a type H agent to zero, and pursue the consequences of this fact on the menu.

(a) The expected revenue is

$$\mathbb{E}[X(e)] = \left(\frac{1+e}{6}\right)(11+25) = 6(1+e).$$

Hence $\mathbb{E}'[X(e)] = 6$.

Optimal wage offers involve fixed wages:

$$\begin{aligned}\sqrt{w_L} &= e+2 \Leftrightarrow \bar{w}_L(e) = (e+2)^2 \\ \sqrt{w_H} &= 2e+2 \Leftrightarrow \bar{w}_H(e) = 4\,(e+1)^2\end{aligned}$$

Hence

$$\bar{w}_L'(e) = 2\,(e+2)\,, \;\; \bar{w}_H'(e) = 8\,(e+1)$$

Optimal effort e_τ solves

$$\max_{e\in[0,1]} \mathbb{E}[X(e)] - w_\tau(e)$$

Thus,

$$\begin{aligned}\mathbb{E}'[X(e)] - w_L'(e) &= 6 - 2\,(e+2) = 2 - 2e = 0 \Leftrightarrow e_L = 1. \\ \mathbb{E}'[X(e)] - w_H'(e) &= 6 - 8\,(e+1) < 0 \Leftrightarrow e_H = 0\end{aligned}$$

Therefore the optimal contracts are $(e_L, \bar{w}_L(e_L)) = (1,9)$ *to the Agent of type* L*, and* $(e_H, \bar{w}_H(e_H)) = (0,4)$ *to the Agent of type* H*.*

(b) Since the optimal contract of the high cost agent involves no effort, the contract $(e_H, \bar{w}_H(0)) = (0, 4)$ remains optimal if there is moral hazard. In order to identify the optimal contract to offer to the low cost agent, we see that $(p_1(0), p_2(0), p_3(0)) = (2/3, 1/6, 1/6)$ and $(p_1(1), p_2(1), p_3(1)) = (1/3, 1/3, 1/3)$, and therefore the likelihood ratios, $l_i = p_i(0)/p_i(1)$, are $(l_1, l_2, l_3) = (2, 1/2, 1/2)$. Hence $l_2 = l_3$ implies $w_2 = w_3 := w_{23}$. Therefore the optimal wage offer $W = (w_1, w_2, w_3)$ that induces and agent L to exert the effort $e = 1$ must satisfy the the system

$$\begin{aligned} \frac{1}{u'(w_1)} &= \lambda + \mu(1-2) \Longleftrightarrow 2\sqrt{w_1} = \lambda - \mu \\ \frac{1}{u'(w_{23})} &= \lambda + \mu\left(1 - \frac{1}{2}\right) \Longleftrightarrow 2\sqrt{w_{23}} = \lambda + \frac{\mu}{2} \end{aligned}$$

In addition, the offer to the τ type agent must satisfy the incentive and participation constraints

$$\begin{aligned} \mathbb{E}[u(W(1)] - 1 &= \mathbb{E}[u(W(0)] \Leftrightarrow \frac{1}{3}\sqrt{w_1} + \frac{2}{3}\sqrt{w_{23}} - 1 = \frac{2}{3}\sqrt{w_1} + \frac{2}{6}\sqrt{w_{23}} \\ \mathbb{E}[u(W(1)] &= 1 + \underline{u} \Leftrightarrow \frac{1}{3}\sqrt{w_1} + \frac{2}{3}\sqrt{w_{23}} = 3 \end{aligned}$$

Solving the system we get $W_L = (1, 16, 16)$. The Principal's profits for this contract is

$$\mathbb{E}[X(1)] - \mathbb{E}[W_L(1)] = 6(1+1) - \frac{1}{3}(1) - \frac{2}{3}(16) = 1.$$

Since the optimal contract for which agents are willing to exert effort $e = 0$ is the fixed wage $w = 4$ for both types, and in either case the profit is

$$\mathbb{E}[X(1)] - \bar{w}_H(0) = 6(1+0) - 4 = 2.$$

then with moral hazard the optimal contract for both types is $(e, w) = (0, 4)$.

(c) The Principal may offer the single contract $(e_L, \bar{w}_L(e_L)) = (1, 9)$, which only L type agents accept, leading to an expected profit

$$\frac{1}{2}\left(\mathbb{E}[X(e_L)] - \bar{w}_L(e_L)\right) = \frac{1}{2}(6(1+1) - 9) = \frac{3}{2}.$$

Alternatively, the Principal may design a menu of contracts that warrants that agents of both types will accept. Since the optimal contract for the type H agent involves no effort the information rents of low cost type agent are zero as well. Hence the menu identified in part (a) is incentive compatible, and therefore is optimal.

Exercise 3. Ann (A), Bob (B) and Conrad (C) share an apartment and must decide the quality of their internet service, $x \in \mathbb{R}_+$. Their preferences are described by utility functions of the form $u_i(x,t) = -t + \alpha_i\sqrt{x}$, where t denotes the amount s/he pays for the service, and $(\alpha_A, \alpha_B, \alpha_C) = (5, 10, 15)$. The internet service provider charges $10x$ for a service of quality x. Assume that all this information is common knowledge to A, B and C.

(c) (10 points) Calculate the Lindahl equilibrium allocation.

(b) (10 points) Calculate the quality of the apartment's internet service and the amount paid by each agent if it is to be paid via voluntary contributions. Is this allocation Pareto optimal?

(c) (10 points) Determine the result of the following voting procedure: A, B and C will simultaneously vote for the quality of service they wish $v_i \geq 0$ with the proviso that they will share the cost equally, and that the apartment's internet quality will be set to $v = (v_A + v_B + v_C)/3$.

Comment. As you know, it is not uncommon that a scheme for deciding the provision of a public good leads to a game with no interior equilibria. This is the case in parts (b) and (c).

Solution

(a) In this setting a Lindahl equilibrium is a Pareto optimal allocation $(x^*, -p_1x, -p_2x, -p_3x)$ *such that* x^* *solves the problem*

$$\max_{x\geq 0} -p_i x + \alpha_i\sqrt{x}$$

for $\alpha_i \in \{5, 10, 15\}$. *The solution to agent* i*'s problem satisfies*

$$\frac{\alpha_i}{2\sqrt{x}} = p_i.$$

Thus,

$$\frac{\alpha_A}{2\sqrt{x}} + \frac{\alpha_B}{2\sqrt{x}} + \frac{\alpha_C}{2\sqrt{x}} = p_1 + p_2 + p_3 \Leftrightarrow x^* = \left(\frac{15}{p_1 + p_2 + p_3}\right)^2.$$

Pareto optimality requires, in addition, that

$$p_1 + p_2 + p_3 = 10.$$

Hence

$$x^* = \left(\frac{15}{10}\right)^2 = \frac{9}{4}.$$

and therefore

$$p_i = \frac{\alpha_i}{2\sqrt{\frac{9}{4}}} = \frac{\alpha_i}{3},$$

that is, $p_A = 5/3$ *euros,* $p_B = 10/3$ *euros, and* $p_C = 15/3$ *euros.*

(b) Under voluntary contributions each agent solves the problem

$$\max_{z_i \geq 0} -z_i + \alpha_i \sqrt{\frac{z_{-i} + z_i}{10}},$$

where z_{-i} *is the sum of the contributions of the other agents. Hence*

$$z_i = \max\{\frac{\alpha_i^2}{40} - z_{-i}, 0\}.$$

Therefore, the equilibrium contributions are $(z_A^*, z_B^*, z_C^*) = (0, 0, 15^2/40)$, *and the internet quality under this scheme is* $x^{vc} = (z_A^* + z_B^* + z_C^*)/10 = 9/16$.

(c) In an interior equilibrium of this game, the strategy of an agent v_i *solves the problem*

$$\max_{v_i \geq 0} - (10) \frac{v_{-i} + v_i}{3} + \alpha_i \sqrt{\frac{v_{-i} + v_i}{3}}.$$

If we denote

$$z_i = \frac{10}{3} v_i, \; z_{-i} = \frac{10}{3} v_{-i},$$

we may write this problem as

$$\max_{z_i \geq 0} z_{-i} + z_i + \alpha_i \sqrt{\frac{z_{-i} + z_i}{10}}.$$

Since z_{-i} *is a given constant (to agent* i*), the solution to this problem, and hence the equilibrium of the game, is that given in part (b),* $(z_A^*, z_B^*, z_C^*) = (0, 0, 15^2/40)$, *which translates to the equilibrium votes*

$$(v_A^*, v_B^*, v_C^*) = (0 \left(\frac{3}{10}\right), 0 \left(\frac{3}{10}\right), \frac{15^2}{40} \left(\frac{3}{10}\right)) = (0, 0, \frac{27}{16}).$$

Of course, the level of internet quality service is the same as under voluntary contribution. However, unlike in that case agents share the cost equally.