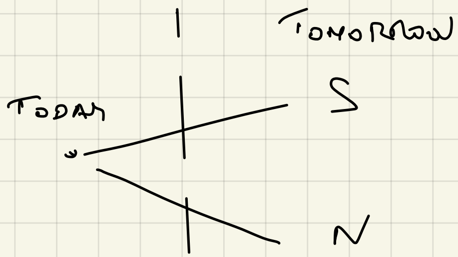
3. Andy's income this year is €20. He is retiring next year, and his income depends on whether Social Security remains financially sound (S), or it goes bankrupt (N). Andy's income in state S will be €20, whereas it will be only €10 in state N (the amount paid by his private pension plan). Beth's income this year is also €20, and her income next year depends on the state of nature: in state S Beth will have to contribute to SS, leaving her with an income of only €10; in state N Beth's income is €20. Andy's preference is represented by the utility function $u^A(x, y_S, y_N) = x + 5\ln y_S + 6\ln y_N$, and Beth's by the function $u^B(x, y_S, y_N) = x + 10\ln y_S + 3\ln y_N$, where x denotes the individual's spending this year, y_S denotes spending next year in state S, and y_N denotes spending next year in state N. Income is perishable; i.e., one unit of income dated this year is worthless next year.



$$(\bar{x}^A, \bar{y}_S^A, \bar{y}_N^A) = (20; 20, 10)$$
$$(\bar{x}^B, \bar{y}_S^B, \bar{y}_N^B) = (20; 10, 20)$$

(a) Determine the competitive allocation and prices assuming that there markets for all goods.

NOTATION: $p = (1, p_S, p_N)$. BC: $x + p_S y_S + p_N y_N = \bar{x} + p_S \bar{y}_S + p_N \bar{y}_N$

DEMANDS: $MRS^i_{x y_S} = \frac{y_S}{\alpha_i} = \frac{1}{p_S}$, $MRS^i_{x y_N} = \frac{y_N}{\beta_i} = \frac{1}{p_N}$.

$(\alpha_A, \beta_A) = (5, 6)$

$(\alpha_B, \beta_B) = (10, 3)$

ANDY: $y^A_S(p) = \frac{5}{p_S}$, $y^A_N(p) = \frac{6}{p_N}$

BETH: $y^B_S(p) = \frac{10}{p_S}$, $y^B_N(p) = \frac{3}{p_N}$.

MARKET CLEARING

$$\frac{5}{p_S} + \frac{10}{p_S} = 30 \implies p_S^* = \frac{1}{2}$$

$$\frac{6}{p_N} + \frac{3}{p_N} = 30 \implies p_N^* = \frac{3}{10}$$

HENCE: $y^A_S(p^*) = 10$, $y^A_N(p^*) = 20$, $y^B_S(p^*) = 20$, $y^B_N(p^*) = 10$

$x^A(p^*) = 20 + \frac{1}{2}(20-10) + \frac{3}{10}(10-20) = 22$

$x^B(p^*) = 20 + \frac{1}{2}(10-20) + \frac{3}{10}(20-10) = 18$

CE ALLOCATION: $[(22, 10, 20), (18, 20, 10)]$

(b) Suppose that only spot markets and a credit market exist. Is the equilibrium allocation Pareto efficient?

NOTATION. $\hat{p}_1 = \hat{p}_S = \hat{p}_N = 1$, $\underline{r}$: INTEREST RATE

CONSUMERS' BUDGET CONSTRAINTS:

$$x + b = \bar{x}$$

$$y_S = \bar{y}_S + (1+r)b$$

$$y_N = \bar{y}_N + (1+r)b$$

$$MRS^A_{y_S,y_N} = \frac{5/y^A_S}{6/y^A_N} = \frac{5}{6}\frac{y^A_N}{y^A_S} = \frac{5}{6}\frac{10+(1+r)b^A}{20+(1+r)b^A}.$$

$$MRS^B_{y_S,y_N} = \frac{10/y^B_S}{3/y^B_N} = \frac{10}{3}\frac{y^B_N}{y^B_S} = \frac{10}{3}\frac{20+(1+r)b^B}{10+(1+r)b^B}.$$

CLEARLY, IN THE CE $\boxed{y^i_S > 0}$, $\boxed{y^i_N > 0}$ $\forall\ i \in \{A, B\}$.

(WHY? IF EITHER $y^i_S = 0$ OR $y^i_N = 0$ CONSUMER i IS WORSE OFF IN THE CE THAN IN THE INITIAL ALLOCATION.)

ALSO: $b^A(r^*) = -b^B(r^*)$.

PARETO OPTIMALITY WOULD REQUIRE:

$$MRS^A_{y_S,y_N} = MRS^B_{y_S,y_N}$$

USING $(1+r^*)\, b^A(r^*) = -(1+r^*)\, b^B(r^*) := x$, WE MAY WRITE THIS CONDITION AS

$$\frac{5}{6}\,\frac{10+x}{20+x} = \frac{10}{3}\,\frac{20-x}{10-x}.$$

THE SOLUTIONS TO THIS EQUATION ARE $x^* = \pm 10\sqrt{5}$.

HOWEVER,

IF $x = 10\sqrt{5} \Rightarrow \boxed{y_N^B = 10 - 10\sqrt{5} < 0}$, WHICH IS A CONTRADICTION

IF $x = -10\sqrt{5} \Rightarrow \boxed{y_S^A = 10 - 10\sqrt{5} < 0}$, WHICH IS A CONTRADICTION

HENCE THE CE ALLOCATION IS <u>NOT</u> PARETO OPTIMAL.

(c) Assume now that there are only spot markets, and the only securities are shares in the firm Gamma Technologies and shares in the firm Delta Insurance. Each share of Gamma will yield €2 in state S and €1 in N. Each share of Delta will yield €1 in state S and €2 in N. Determine the equilibrium security prices and Andy's and Beth's holdings of securities. What portfolio would one have to hold in order to guarantee oneself a return of €1 next year whether the state is S or N? What would be the cost of that portfolio? What would you say is the interest rate?

$r_D = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $r_G = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$; $p_1 = p_S = p_N = 1.$ q_D, q_G CE: $-z_G^A = z_D^A = z_G^B = -z_D^B = 10$

IN THIS SETTING A CONSUMER'S BUDGET CONSTRAINTS ARE:

(THIS YEAR) $x + q_D z_D + q_G z_B = \bar{x}$

(NEXT YEAR S) $y_S = \bar{y}_S + z_D + 2z_G$

(NEXT YEAR N) $y_N = \bar{y}_N + 2z_D + z_G$

SOLVING FOR z_D, z_G IN NEXT YEAR CONSTRAINTS WE GET

$$z_D = \frac{1}{3}(\bar{y}_S - y_S) - \frac{2}{3}(\bar{y}_N - y_N)$$
$$z_G = -\frac{2}{3}(\bar{y}_S - y_S) + \frac{1}{3}(\bar{y}_N - y_N)$$

SUBSTITUTING THESE VALUES IN THIS YEAR'S BUDGET CONSTRAINT WE GET :

$$x + \left(\frac{2}{3} q_D - \frac{1}{3} q_G\right) y_S + \left(\frac{2}{3} q_G - \frac{1}{3} q_D\right) y_N = \bar{x} + \left(\frac{2}{3} q_D - \frac{1}{3} q_G\right) \bar{y}_S + \left(\frac{2}{3} q_G - \frac{1}{3} q_D\right) \bar{y}_N$$

COMPARING THIS BUDGET CONSTRAINT WITH THAT OF PART (a) OF THE EXERCISE, WE SEE THAT A CONSUMER'S PROBLEM IS THE SAME, AND THEREFORE IN EQUILIBRIUM THE RELATE MARKET VALUES OF GOODS MUST BE THE SAME. THAT IS:

$$\left.\begin{aligned} \frac{3}{2} q_D^* - \frac{1}{3} q_G^* &= p_S^* = \frac{1}{2} \\ \frac{3}{2} q_G^* - \frac{1}{3} q_D^* &= p_N^* = \frac{3}{10} \end{aligned}\right\} \iff \begin{aligned} q_D^* &= \frac{11}{10} \\ q_G^* &= \frac{13}{10}. \end{aligned}$$

As for the <u>implicit interest rate</u>, note that giving up 1 unit of consumption today one can buy the portfolio

$$\begin{pmatrix} z_C \\ z_D \end{pmatrix} = \begin{pmatrix} z \\ z \end{pmatrix}, \quad \text{where} \quad 1 = q_D z + q_C z \Leftrightarrow z = \frac{1}{q_D + q_C} = \frac{10}{24}$$

which yields

$$\begin{array}{c} S \\ N \end{array} \begin{pmatrix} 2z_C + z_D \\ z_C + 2z_D \end{pmatrix} = \begin{pmatrix} 30/24 \\ 30/24 \end{pmatrix}$$

units of consumption in each state. Hence

$$\frac{30}{24} = 1 + r \Leftrightarrow \boxed{r = 1/4}$$

is the "implicit" interest rate.

[redacted] A pure exchange economy operates over two dates, today and tomorrow. The state of nature tomorrow is uncertain and can be either sunny (S) or cloudy (C). There is a single perishable good, consumption, and two consumers. The consumers' preferences for consumption today (x), consumption tomorrow if sunny (y_S), and consumption tomorrow if cloudy (y_C) are represented by the utility functions $u_1(x, y_S, y_C) = x(2y_S + y_C)$ and $u_2(x, y_S, y_C) = x(y_S + 2y_C)$, respectively, and both are have the initial endowments $(4, 2, 2)$. There are no contingent markets, but there are spot markets for consumption at each date, as well as a credit market and a market for a security both operating today. The security pays 1 unit of the good in tomorrow if sunny and nothing otherwise. Verify that the security price $q^* = 1$ and interest rate $r^* = -1/2$ lead to a competitive equilibrium, and calculate the corresponding allocation. [redacted]

Budget Constraints

$$x = 4 - (1)s + b$$

$$y_s = 2 + s - (1 - 1/2)b \implies s = y_s + \frac{b}{2} - 2$$

$$y_c = 2 - (1 - 1/2)b \implies b = 4 - 2y_c$$

$$\implies x = 4 - \left(y_s + \cancel{2} - y_c - \cancel{2}\right) + \left(4 - 2y_c\right)$$

$$x + y_s + y_c = 4 + 2 + 2 \qquad \left(P_x^* = P_{y_s}^* = P_{y_c}^* = 1 \ ?\right)$$

1

$$MRS^1_{y_s, y_c} = 2$$

$$\implies y_{1c}^* = 0 \implies 2 - \frac{b_1}{2} = 0 \implies b_1^* = 4,$$

and

$$\max_s (4 - s + 4)(\cancel{2} + s - \cancel{2})$$
$$= (8 - s)s$$

F.O.C. $8 - 2s = 0 \implies s_1^* = 4$

2

$$MRS^2_{y_s, y_c} = \frac{1}{2} \implies y_{2s}^* = 0$$

$$\implies s_2 = \frac{b}{2} - 2$$

$$\implies \max_b \left(4 - \left(\frac{b}{2} - 2\right) + b\right)\left(2 - \frac{b}{2}\right)$$
$$\left(6 + \frac{b}{2}\right)\left(2 - \frac{b}{2}\right)$$

F.O.C. $\cancel{\frac{1}{2}}\left(2 - \frac{b}{2}\right) - \cancel{\frac{1}{2}}\left(6 + \frac{b}{2}\right) = 0$

$$b_2^* = -4$$

$$s_2^* = -\frac{4}{2} - 2 = -4$$

Hence, $b_1^* + b_2^* = 0$, $s_1^* + s_2^* = 0$, so

$q^* = 1$, $r^* = -1/2$ are indeed a CE prices.

The CE allocation is

$x_1^* = 4 - s_1^* + b_1^* = 4$

$y_{1s}^* = 2 + s_1^* - \frac{b_1^*}{2} = 4$

$y_{1c} = 0$

$x_2^* = 4 - s_2^* + b_2^* = 4$

$y_{2s}^* = 0$

$y_{2c} = 2 - \frac{b_2^*}{2} = 4$